Cartesian-Carrier Symmetric Monoidal Structures on Games

A profinite moduli and its closed structures

ChatGPT

Version 0.1 — 28 August 2026

Abstract

Let Game be the category of finitely branching, finite-time impartial games and locally surjective graph maps, and let $U\colon \mathsf{Game} \to \mathsf{Set}$ be the forgetful functor. We describe symmetric monoidal structures on Game whose tensor carrier is cartesian product and whose coherence maps lie over the standard cartesian coherence bijections. The datum is a natural rooted transition law on pairs of finite rooted games, subject to finite unit, symmetry, and associativity equations. This presents the collection of such structures as an effectively closed profinite space. It contains continuum many concrete points. We give a uniform cofree-game construction of the corresponding internal homs, isolate the two coordinate-local points (Conway and selective sum), and derive exact associativity criteria for two large families governed by height drops and path lengths. Strong structures with the same product carrier transport to this moduli problem, while genuinely lax unital structures are not classified here. This is an AI-generated review draft: the cartesian-carrier classification is the claimed result; the unrestricted lax problem and several quotient-by-equivalence questions remain open.

Review status. This manuscript was reconstructed from a mathematical research conversation after its earlier generated files became unavailable. It deliberately narrows the scope to cartesian-carrier structures. It is submitted for private mathematical review, not presented as an independently verified final theorem.

1 Scope and relation to the motivating problem

Hora's category of games is a category of recursive $\mathcal{P}_f$-coalgebras with a remarkably rigid forgetful functor [1]. His Questions 5.5 and 5.6 ask for the possible closed monoidal structures and for the additional possibilities visible to lax monoidal forgetful functors. Three familiar operations already show that the answer cannot be a short uniqueness theorem: Conway sum allows a move in exactly one coordinate, selective sum also allows a simultaneous move, and conjunctive sum requires both coordinates to move (the last is non-unital inside finite-time games).

The purpose of this note is to separate a tractable and already very large moduli problem from the genuinely lax problem. We impose

$$U(X \otimes Y) = UX \times UY, \qquad U\mathbb{I} = 1$$

as equalities, and require all coherence maps to have the standard cartesian coherence bijections as their underlying functions. The problem is then not to choose carriers but to choose the option relation on each cartesian product, naturally and coherently.

The main result is a finite-data classification: the entire operation is determined by what it does at the roots of pairs of finite rooted games. The answer is not a finite list.

It is an effectively closed profinite space, and even a simple height-sensitive stratum has the cardinality of the continuum. This is analogous in spirit to classification results for closed structures on graphs [2, 3], but the recursive and locally finite nature of games gives a different parameter space.

What is not claimed. We do not classify arbitrary lax monoidal structures on U, nor do we prove that the continuum many cartesian-carrier points remain pairwise inequivalent after quotienting by every monoidal autoequivalence of Game. We also do not claim new Grundy formulae for operations already known as Ryuo Nim or mixed-radix Nim [5, 6].

2 Games, roots, and finite pieces

Definition 2.1. A *game* is a directed graph $X = (UX, \to_X)$ such that every vertex has finitely many outgoing edges and no infinite directed path starts at any vertex. Write

$$\mathrm{Opt}_X(x) = \{x' \in UX : x \to_X x'\}.$$

A *game morphism* $f\colon X \to Y$ is a function satisfying the exact option equation

$$f[\mathrm{Opt}_X(x)] = \mathrm{Opt}_Y(fx) \qquad (x \in UX).$$

Equivalently, it is a graph map with the path-lifting property. The resulting category is denoted by Game, and $U\colon \mathsf{Game} \to \mathsf{Set}$ forgets the option relation.

For $x \in X$, let $\langle x\rangle_X$ be the generated subgame consisting of all descendants of x, with distinguished root x. The following elementary fact is the reason finite rooted data suffice.

Lemma 2.2 (finite rooted pieces). *For every $x \in X$, the rooted game $\langle x\rangle_X$ is finite. Every game is the filtered union of its finite subgames.*

Proof. The tree of finite paths starting at x is finitely branching. If it had infinitely many nodes, König's lemma would give an infinite branch, contrary to the finite-time condition. Thus only finitely many paths, and hence only finitely many descendants, occur. Finite unions of finitely generated subgames give the filtered presentation. □

The *height* $h_X(x)$ is the largest length of a path starting at x. It is finite, and $x \to x'$ implies $h_X(x') < h_X(x)$. Notice that a game may have unbounded heights globally.

Let $\mathsf{Game}_{\mathrm{fr}}$ be the essentially small category whose objects are pairs (A, a) with A a finite game generated by a, and whose maps preserve the root. A rooted map is automatically surjective on each layer of options, although it may identify positions.

3 Rooted transition laws

Definition 3.1 (rooted transition law). A *rooted transition law* is a family of finite subsets

$$\tau(A, a; B, b) \subseteq UA \times UB$$

for pairs of finite rooted games, satisfying:

(T1) **Progress:** on the finite set $UA \times UB$, form the relation

$$(u, v) \longrightarrow (u', v') \quad \iff \quad (u', v') \in \tau(\langle u\rangle_A, u; \langle v\rangle_B, v).$$

This relation is acyclic. In particular, (a, b) is not a root option of itself.

(T2) **Rooted naturality:** for rooted game maps $f\colon (A,a) \to (A',a')$ and $g\colon (B,b) \to (B',b')$,

$$(f \times g)[\tau(A,a;B,b)] = \tau(A',a';B',b').$$

The law extends to arbitrary games by the formula

$$\mathrm{Opt}_{X \otimes_\tau Y}(x,y) = \tau(\langle x\rangle_X, x; \langle y\rangle_Y, y). \tag{1}$$

The carrier is $UX \times UY$. The option set is finite. An infinite path from (x,y) would remain inside the finite set $U\langle x\rangle_X \times U\langle y\rangle_Y$, contradicting progress. Naturality is exactly the option equation for $f \times g$.

There are three further finite equations.

Definition 3.2 (admissibility)**.** A rooted transition law is *admissible* when it satisfies:

(C1) **Unit:** for the one-point zero game $1 = \{*\}$,

$$\tau(A,a;1,*) = \mathrm{Opt}_A(a) \times \{*\}, \qquad \tau(1,*;A,a) = \{*\} \times \mathrm{Opt}_A(a).$$

(C2) **Symmetry:** $\tau(B,b;A,a)$ is obtained from $\tau(A,a;B,b)$ by interchanging the coordinates.

(C3) **Associativity:** construct $A \otimes_\tau B$ and $B \otimes_\tau C$ using (1), and let P and Q be the subgames generated by (a,b) and (b,c), respectively. Under the canonical embeddings into $UA \times UB \times UC$, the two root-option sets

$$\tau(P,(a,b);C,c), \qquad \tau(A,a;Q,(b,c))$$

are equal.

All terms in these equations are finite. In particular, admissibility is decidable for any finite collection of coordinates of a proposed law.

Theorem 3.3 (classification)**.** *Admissible rooted transition laws are in bijection with symmetric monoidal structures* $(\otimes, 1)$ *on* Game *for which the tensor and unit carriers are* $UX \times UY$ *and* 1*, and the coherence maps have the standard cartesian coherence bijections as their underlying functions. The correspondence is given by* (1) *in one direction and by taking root options in the other.*

Proof. Given an admissible law, progress makes (1) a finitely branching finite-time relation. Rooted naturality says that $f \times g$ is a game morphism whenever f and g are. At a triple (x,y,z), apply (C3) to the rooted games generated by x,y,z. The generated parts P,Q in (C3) are exactly the finite pieces used by (1) for the two outer products. Hence the reassociation function is exact at every triple. The same argument with (C1) and (C2) treats unit and swap. Their underlying functions are the standard cartesian coherence maps, so the coherence diagrams commute.

Conversely, let $\otimes$ be such a cartesian-carrier structure and define

$$\tau_\otimes(A,a;B,b) = \mathrm{Opt}_{A \otimes B}(a,b).$$

Finite branching gives a finite subset. Tensoring the inclusions of descendant subgames is exact, so the reconstruction relation at every descendant root is the original option relation in $A \otimes B$; its finite-time condition gives progress. Because the carrier is the product and the tensor acts on maps by the product function, exactness of game morphisms gives rooted naturality. For (C3), tensor the inclusions $P \hookrightarrow A \otimes B$ and $Q \hookrightarrow B \otimes C$ with the remaining identity maps. Exactness, followed by the standard associativity map, identifies the two displayed root-option sets. Unit and symmetry similarly give (C1) and (C2).

It remains to compare the reconstructed relation at an arbitrary pair (x, y). The inclusions of the generated finite subgames $\langle x \rangle \hookrightarrow X$ and $\langle y \rangle \hookrightarrow Y$ are game morphisms. Their product is exact at (x, y), so the option set in $X \otimes Y$ is the image of the root-option set in the finite tensor. This is precisely (1). The two constructions are therefore inverse. $\square$

Remark 3.4 (rank certificates). Progress is a finite acyclicity condition, so every finite coordinate admits some decreasing rank. Requiring the particular rank $h_A + h_B$ would exclude valid product-carrier structures unnecessarily. The concrete height-drop and path-length families below do preserve this particular rank, which gives a convenient direct certificate for them.

4 The profinite moduli

Fix a computable skeleton with one representative of every isomorphism class of finite rooted game. There are countably many, and tables on isomorphic copies are transported by rooted naturality. For each ordered pair $((A, a), (B, b))$, the possible root-option sets form the finite discrete space

$$2^{UA \times UB}.$$

Their countable product $\mathcal{T}$ is compact, metrizable, totally disconnected, and profinite.

Theorem 4.1 (profinite presentation)**.** *The admissible laws form an effectively closed subspace $\mathcal{M}_{\mathrm{cart}} \subseteq \mathcal{T}$.*

Proof. A failure of progress, naturality, unit, symmetry, or associativity is witnessed on finitely many finite rooted games and finitely many entries of their transition tables. Hence the complement of each condition is a union of basic open cylinders, and the admissible locus is closed. Finite games, rooted maps, and the required finite equations can be enumerated effectively, so the list of forbidden cylinders is recursively enumerable. $\square$

The word "classification" in Theorem 3.3 therefore means an explicit profinite moduli presentation, not a finite list of familiar operations.

5 Two concrete strata

5.1 Coordinate-local laws

Call a law coordinate-local if the decision to include a pair of component options depends only on whether the left coordinate moves, the right coordinate moves, or both, and not on the surrounding games or positions.

Proposition 5.1. *There are exactly two symmetric unital coordinate-local laws: Conway sum and selective sum.*

Proof. The unit equations force every one-coordinate move and forbid a stationary move. Symmetry leaves one binary choice: either no pair of simultaneous component options is included, giving Conway sum, or every such pair is included, giving selective sum. Both choices are associative. $\square$

5.2 Height-drop laws

We first record a consequence of the exact option equation. If $f : X \to Y$ is a game morphism, then

$$h_Y(fx) = h_X(x)$$

for every x. Indeed, induction on height and $f[\mathrm{Opt}_X(x)] = \mathrm{Opt}_Y(fx)$ identify the two maxima defining height. Thus f also preserves the height drop along every edge.

For an edge $x \to x'$, write $\delta(x, x') = h(x) - h(x') > 0$. Given $S \subseteq \mathbb{N}_{>0}$, define $\otimes_S$ by taking all one-coordinate moves and also the simultaneous moves

$$(x, y) \longrightarrow (x', y') \quad \text{when} \quad x \to x',\ y \to y',\ \delta(x, x'), \delta(y, y') \in S.$$

The height identity above makes this rule natural under game morphisms. The maximum path length in $X \otimes_S Y$ is $h_X + h_Y$, since all one-coordinate moves remain available.

Theorem 5.2 (height-drop criterion)**.** *The operation $\otimes_S$ is associative if and only if*

$$a + b \in S \iff b + c \in S \qquad (a, b, c \in S). \tag{2}$$

Equivalently, either S is sum-free or $S + S \subseteq S$.

Proof. All triple option types involving at most two moving coordinates agree under the two parenthesizations. Suppose all three coordinates move with height drops $a, b, c \in S$. In $(X \otimes_S Y) \otimes_S Z$, the first two moves form an edge whose height drop is $a + b$; the outer simultaneous move is present exactly when $a + b \in S$. The other parenthesization is present exactly when $b + c \in S$. Thus (2) is sufficient.

It is also necessary. To realize a prescribed $d \geq 1$, take a root with a direct edge to a terminal position and a separate branch of length d. The direct edge has height drop d. Three such finite witnesses distinguish the two triple option sets whenever (2) fails.

Finally, for fixed $b \in S$, condition (2) says that all sums $a + b$ with $a \in S$ are either in S or outside S. The choice cannot depend on b: if b and d made opposite choices, the common sum $b + d$ would be both in and outside S. Hence either every sum of two elements of S lies in S, or none does. $\square$

Corollary 5.3. *There are continuum many distinct cartesian-carrier symmetric monoidal structures on the fixed category* Game.

Proof. Every subset of the odd positive integers is sum-free, hence gives an associative law. Distinct subsets are distinguished by a pair of finite games having edges with a prescribed height drop. There are $2^{\aleph_0}$ such subsets, while the ambient countable product $\mathcal{T}$ has cardinality at most $2^{\aleph_0}$. $\square$

This counts concrete structures with cartesian carriers before quotienting by arbitrary monoidal autoequivalences.

5.3 Path-length laws

Let $D \subseteq \mathbb{N}^2 \setminus \{(0, 0)\}$. Put an edge $(x, y) \to (x', y')$ when there are directed paths of lengths m, n from x, y to x', y' and $(m, n) \in D$. To obtain a symmetric unital operation, require D to be invariant under coordinate swap and require its intersections with the axes to be exactly $\{(1, 0), (0, 1)\}$. The exclusion of $(0, 0)$ makes the sum of component heights decrease at every move. The axis condition is precisely the canonical unit law, and swap-invariance is precisely symmetry.

For later use, an induction on n in the exact option equation gives

$$f[\{x' : x \longrightarrow^n x'\}] = \{y' : fx \longrightarrow^n y'\}$$

for every game morphism f and $n \geq 0$. Hence path-length laws are rooted natural.

Write D^{+r} for the r-fold Minkowski sum, with $D^{+0} = \{(0,0)\}$. Both parenthesizations of a triple operation are again path-length operations, now on triples.

Proposition 5.4 (path-length criterion)**.** *The path-length law associated with D is associative if and only if*

$$\bigcup_{(r,n)\in D} D^{+r} \times \{n\} = \bigcup_{(m,r)\in D} \{m\} \times D^{+r}$$

as subsets of $\mathbb{N}^3$.

Proof. A path of length r in the inner product is a concatenation of r product edges. The pair of total component lengths therefore belongs to D^{+r}. This gives the left-hand set for $(X \otimes_D Y) \otimes_D Z$ and the right-hand set for $X \otimes_D (Y \otimes_D Z)$. Equality is sufficient. Conversely, three sufficiently long chains separate every bounded triple of lengths, so a triple in the symmetric difference witnesses failure of associativity. □

6 A uniform internal-hom construction

Hora proves abstract closedness for Conway sum and asks for an explicit internal hom [1]. A separate private draft, `pre-q2ygeh`, gives such a construction for Conway sum and explicitly leaves arbitrary product-carrier liftings open. A general law requires a compatibility equation adapted to its own transition table; the Conway formula cannot simply be cited unchanged.

We use the cofree game. Since U is comonadic it has a right adjoint $R\colon \mathsf{Set} \to \mathsf{Game}$. A vertex of $R(S)$ carries a label in S, together with its possible future labelled behaviors. For games G, H, take $S = (UH)^{UG}$. The root label of a vertex $w \in R(S)$ is a function $\ell_w\colon UG \to UH$.

There is a set map

$$UG \times UR(S) \longrightarrow UH, \qquad (g, w) \longmapsto \ell_w(g).$$

It need not be a game morphism on all of $G \otimes R(S)$. Let $[G,H]_\tau$ be the union of all subgames $K \hookrightarrow R(S)$ on which the restricted evaluation map $G \otimes K \to H$ is a game morphism.

Theorem 6.1 (uniform closed structure)**.** *For every admissible rooted transition law τ, the construction above is an internal hom:*

$$\mathsf{Game}(G \otimes_\tau K, H) \cong \mathsf{Game}(K, [G,H]_\tau),$$

naturally in K and H.

Proof. The union in the definition is again a subgame. If a compatible subgame contains w, it contains every descendant of w in $R(S)$, so enlarging by other compatible subgames adds no new options at w. Evaluation therefore remains exact on the union.

Given $p\colon G \otimes K \to H$, curry its underlying function to $UK \to (UH)^{UG}$ and take the cofree transpose $\widehat{p}\colon K \to R((UH)^{UG})$. The image of any game morphism is a subgame, by the exact option equation. Moreover, $\mathrm{id}_G \otimes \widehat{p}$ is a game morphism and its image is

$UG \times \mathrm{im}(\widehat{p})$. Since $p = \mathrm{ev} \circ (\mathrm{id}_G \otimes \widehat{p})$ on underlying sets and both p and $\mathrm{id}_G \otimes \widehat{p}$ are exact, evaluation is exact on that image. Thus $\mathrm{im}(\widehat{p})$ is one of the compatible subgames and $\widehat{p}$ factors through $[G, H]_\tau$.

Conversely, compose a map $K \to [G, H]_\tau$ with the restricted evaluation. It is a game morphism by construction. One round trip is ordinary uncurrying on underlying sets. For the other, the inclusion into $R((UH)^{UG})$ composed with the cofree transpose agrees with the inclusion composed with the original map; uniqueness in the cofree adjunction makes the maps themselves equal. Thus the two constructions are inverse. Naturality follows from the same adjunction and the naturality of evaluation. □

Remark 6.2 (review point). Theorem 6.1 depends on taking the largest compatible subgame, not merely the set of compatible root labels. This descendant-wise condition is essential for path lifting. A fully formal implementation can define the subgame as the union of images of all cofree transposes; this avoids any choice of representatives.

7 Strong and lax boundaries

If U is strong symmetric monoidal and its comparison maps identify $U(X \otimes Y)$ naturally with $UX \times UY$, transport the game relation across those bijections. Mac Lane coherence then yields a cartesian-carrier representative, so Theorem 3.3 describes the strong case up to this transport.

There is also a useful rigidity statement when the carrier has already been fixed to the product.

Lemma 7.1. *Every natural transformation $U \Rightarrow U$ is the identity.*

Proof. Fix a position x. In its finite rooted subgame, replace each root option subgame by two disjoint isomorphic copies and map the result back by the fold map. The common fixed-point set of all automorphisms that swap one pair of copies is just the root. Naturality forces the image of the root under the transformation to lie in this common fixed-point set. Naturality with the fold map therefore fixes x in its generated subgame, and naturality with the inclusion into the original game fixes x there as well. Since x was arbitrary, the transformation is the identity. □

Let $\mu_{X,Y} : UX \times UY \to U(X \otimes Y) = UX \times UY$ be a natural comparison. Fix Y and $y \in UY$. As X varies, the first coordinate of $\mu_{X,Y}(x, y)$ is a natural endomorphism of U, hence is x by Lemma 7.1. Similarly, after fixing X and $x \in UX$, the second coordinate is y. Thus $\mu_{X,Y}$ is the identity, even without a unit. No extra comparison maps appear on a fixed product carrier.

Removing the unit nevertheless changes the transition-law equations: rooted laws without (C1) still give a large non-unital moduli. If one also removes the product-carrier restriction, constant tensor functors already produce a proper class of lax non-unital examples. This is an existence statement, not a classification up to monoidal equivalence.

Finally, a closed tensor preserves colimits separately in each variable, and Lemma 2.2 gives a finite-input density reduction

$$X \otimes Y \cong \operatorname*{colim}_{A \subseteq_{\mathrm{fin}} X, B \subseteq_{\mathrm{fin}} Y} A \otimes B.$$

This reduces a genuinely lax closed problem to its values on finite inputs, but those values need not themselves be finite. It is therefore a finite-input kernel, not a finite classification.

8 Open problems and verification boundary

The cartesian-carrier moduli presentation suggests several precise next questions.

1. Classify genuinely lax unital closed structures, beginning with their finite-input kernels.
2. Determine the quotient of $\mathcal{M}_{\mathrm{cart}}$ by monoidal autoequivalences of Game.
3. Describe the locus for which the induced Bouton quotient is finite or computable. Context separation for selective sum remains conjectural.
4. Compare height-drop points with known selective-compound games. The mixed-radix Grundy formulae in [5, 6] are prior results; the new datum here is their position inside the transition-law moduli.
5. Mechanize the finite rooted coherence equations and independently check the concrete families. Finite computation may verify instances, but it is not a substitute for Theorems 3.3, 5.2, and 6.1.

Authorship and provenance. This reconstructed review draft is displayed under the author name ChatGPT. Ryuya Hora's published paper supplies the category, the motivating questions, the Conway tensor, and abstract closedness, but Hora is not represented as having authored or checked the new classification asserted here.

References

[1] R. Hora, *Games as recursive coalgebras: A categorical view on the Nim-sum*, arXiv:2510.22886, 2025. https://arxiv.org/abs/2510.22886.

[2] K. Kapulkin and N. Kershaw, *Closed symmetric monoidal structures on the category of graphs*, arXiv:2310.00493, 2023. https://arxiv.org/abs/2310.00493.

[3] A. Grenier and C. Kapulkin, *Biclosed monoidal structures on the categories of digraphs and graphs*, arXiv:2511.17739, 2025. https://arxiv.org/abs/2511.17739.

[4] F. Foltz, C. Lair, and G. M. Kelly, Algebraic categories with few monoidal biclosed structures or none, *Journal of Pure and Applied Algebra* 17 (1980), 171–177. https://doi.org/10.1016/0022-4049(80)90082-1.

[5] R. Miyadera, Y. Tokuni, Y. Nakaya, M. Fukui, T. Abuku, and K. Suetsugu, *Ryuo Nim: A Variant of the classical game of Wythoff Nim*, arXiv:1711.01411, 2017. https://arxiv.org/abs/1711.01411.

[6] Y. Irie, *Mixed-radix Nim*, arXiv:1801.00616, 2018. https://arxiv.org/abs/1801.00616.