TOPOS WITH ENOUGH PROJECTIVES

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Abstract. In this paper, we provide a site characterization of topoi with enough projectives. More concretely, we prove that a Grothendieck topos has enough projective objects if and only if it is a sheaf topos over a κ-extensive site for some regular cardinal κ. These observations lead us to generalize the notion of Gaeta topoi to κ-Gaeta topoi. We will see that 2-Gaeta topoi are precisely presheaf topoi.

memo: cite: Collapsed Toposes and Cartesian Closed Varieties by Peter Johnstone

memo: These topoi might be called κ-Gaeta topos. See [Gaeta topos]. For the finite regular cardinal 2, 2-Gaeta topos is presheaf topoi. See also [DECIDABLE OBJECTS AND MOLECULAR TOPOSES]

- The question is asked at [mathoverflow 316588] by Morgan Rogers.
- Ryo Suzuki asks a similar question in the context of internal choice principle in the topos of light condensed sets.

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1. Introduction

2. κ-extensive site and κ-Gaeta topoi

For a (possibly finite) cardinal κ, we mean 'less than κ' by the word 'κ-small.'

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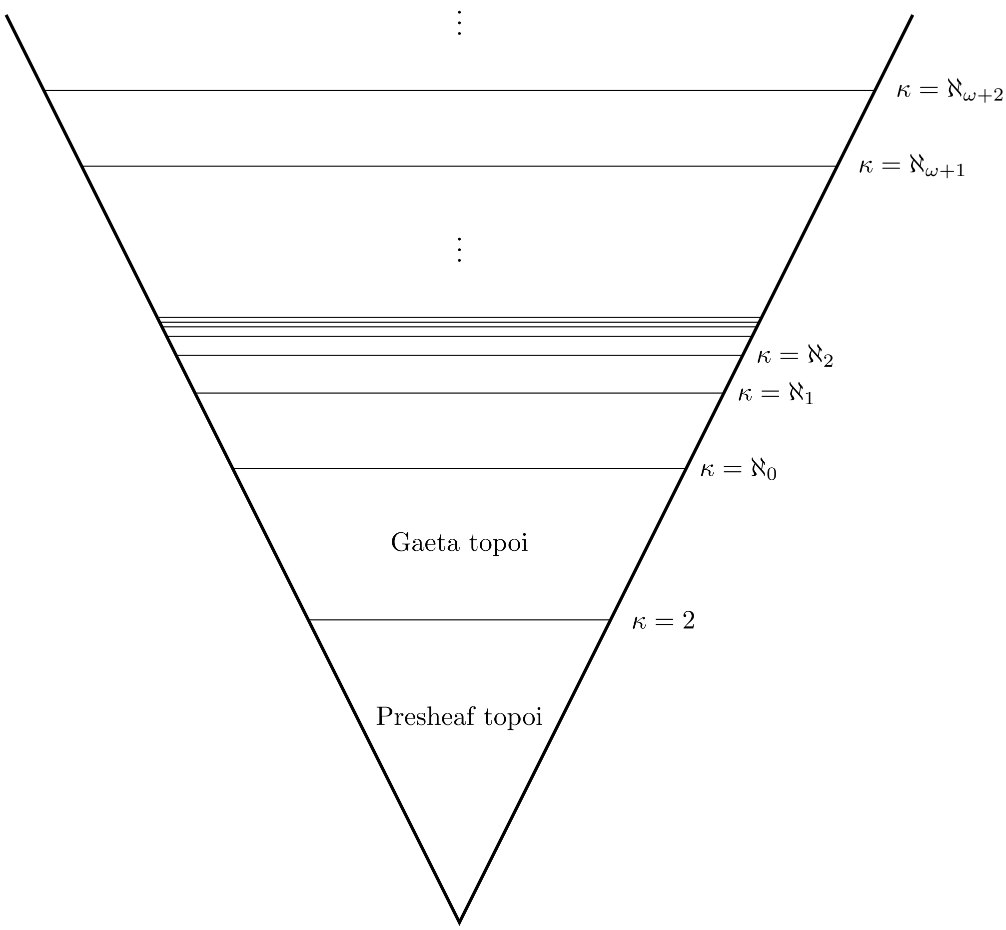

FIGURE 1. Hierarchy of topoi with enough projectives

Definition 2.1. A (possibly finite) cardinal κ is **regular**, if

- $1 \in \kappa$, and
- for any κ-small family of κ-small sets $\{X_i\}_{i\in I}$,
$$|I| < \kappa \text{ and } \forall i \in I\ |X_i| < \kappa$$
then their sum $\coprod_{i\in I} X_i$ is also κ-small.

The sequence of regular cardinals starts with $\kappa = 2, \aleph_0, \aleph_1, \ldots, \aleph_{\omega+1}, \ldots$.
memo: The condition $1 \in \kappa$ is natural, in the sense that [Small sheaves paper]

Definition 2.2. For a regular cardinal κ, a small category $\mathcal{C}$ is said to be κ-extensive, if $\mathcal{C}$ admits

- κ-small coproducts, and
- pullbacks of κ-small coproduct inclusion maps (along arbitrary maps),

and furthermore, κ-small coproducts are

- disjoint, and
- pullback stable.

$\kappa = 2$: A category $\mathcal{C}$ is 2-extensive, if and only if $\mathcal{C}$ has a strict initial object.
$\kappa = \aleph_0$: The $\aleph_0$-extensivity is the usual extensivity as studied in [CLW93].

Notation 2.3. For a regular cardinal κ and a small κ-extensive category $\mathcal{C}$, we write $J_{\kappa\text{-ext}}$ for the collection of sieves S that contain a κ-small coproduct diagram. In other words, a sieve S belongs to $J_{\kappa\text{-ext}}$ if and only if there is a κ-small family $\{X_i \to X\}_{i\in I} \subset S$ such that the canonical map

$$\coprod_{i\in I} X_i \to X$$

is an isomorphism.

(The following lemma is known. For example, memo: [UltracategoriesAppendix] proves this for the cases where $\kappa = \aleph_0$, and $\mathcal{C}$ has pullbacks. We prove the following in a general form.)

Lemma 2.4 (Description of κ-extensive topology)**.** *For a regular cardinal κ and a small κ-extensive category $\mathcal{C}$, the collection $J_{\kappa\text{-ext}}$ is a Grothendieck topology on $\mathcal{C}$.*

Proof. We need to check the three axioms. memo: We need to cite something

Trivial cover: For any object $X \in \mathrm{ob}(\mathcal{C})$, the trivial sieve $\langle \mathrm{id}_X \rangle$ belongs to $J_{\kappa\text{-ext}}$, since $\mathrm{id}_X : X \to X$ is a κ-small coproduct diagram. (We used the assumption $1 \in \kappa$.)
Stability: This follows from the κ-extensivity of $\mathcal{C}$, especially from the condition that κ-small coproducts are pullback stable.
Transitivity: This follows from (the second condition of) the regularity of κ.

□

Definition 2.5. For a regular cardinal κ and a small κ-extensive category $\mathcal{C}$, we call the Grothendieck topology $J_{\kappa\text{-ext}}$ (Theorem 2.3, Theorem 2.4) **the κ-extensive topology** associated to $\mathcal{C}$. We call the site $(\mathcal{C}, J_{\kappa\text{-ext}})$ **the κ-extensive site** associated to $\mathcal{C}$.

Definition 2.6 (κ-Gaeta topos)**.** For a regular cardinal κ, we call a Grothendieck topos $\mathcal{E}$ a **κ-Gaeta topos**, if there exists a small κ-extensive category $\mathcal{C}$ such that $\mathcal{E} \simeq \mathbf{Sh}(\mathcal{C}, J_{\kappa\text{-ext}})$.

Proposition 2.7. *For a regular cardinal κ and a small κ-extensive category $\mathcal{C}$, a presheaf $T : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ is a $J_{\kappa\text{-ext}}$-sheaf if and ony if it sends κ-small coproducts to κ-small products.*

$$T\left(\coprod_{i\in I} X_i\right) \cong \prod_{i\in I} T(X_i)$$

Proof. We first assume that T is a $J_{\kappa\text{-ext}}$-sheaf, and prove that T preserves κ-small coproducts. Since the empty sieve covers the strict initial object $\emptyset \in \mathrm{ob}(\mathcal{C})$, we have $T(\emptyset) = 1$. For a κ-small coproduct diagram $\{X_i \to X\}_{i\in I}$, let S denote the sieve on X generated by the coproduct inclusions. By the κ-extensivity of $\mathcal{C}$ (especially by the disjointness of κ-small coproducts), the sieve $S \rightarrowtail \text{よ}(X)$ (regarded as an object of $\mathbf{PSh}(\mathcal{C})$) is the pushout of

$$\begin{tikzcd}
 & \text{よ}(X_i) \\
\text{よ}(\emptyset) \arrow[ur, tail] \arrow[r, tail] \arrow[dr, tail] \arrow[ddr, tail] & \text{よ}(X_{i'}) \\
 & \text{よ}(X_{i''}) \\
 & \vdots
\end{tikzcd}$$

in the presheaf topos $\mathbf{PSh}(\mathcal{C})$. Thus the glueing condition of $S \rightarrowtail \text{よ}(X)$ states that

$$\begin{tikzcd}
 & T(X_i) \arrow[dr] & \\
T(X) \arrow[ur] \arrow[r] \arrow[dr] \arrow[ddr] & T(X_{i'}) \arrow[r] & T(\emptyset) \\
 & T(X_{i''}) \arrow[ur] & \\
 & \vdots \arrow[uur] &
\end{tikzcd}$$

is a pullback diagram (in **Set**). Thus, $T(\emptyset) \cong 1$ implies $T(X) \cong \prod_{i \in I} T(X_i)$.

Conversely, we assume that T sends κ-small coproducts to κ-small products. We prove that T is a $J_{\kappa\text{-ext}}$-sheaf. Let S be a $J_{\kappa\text{-ext}}$-covering sieve on an object $X \in \mathrm{ob}(\mathcal{C})$, and $\{t_f \in T(Y)\}_{f\colon Y \to X \in S}$ is a S-matching family. By definition of $J_{\kappa\text{-ext}}$, it contains a κ-small coproduct diagram $\{\iota_i \colon X_i \to X\}_{i \in I}$. By the assumption, we obtain $t \in T(X)$ that corresponds to (t_{ι_i}) via the canonical bijection $T(X) \cong \prod_{i \in I} T(X_i)$. In other words, $t \in T(X)$ is the unique elemnt such that $t_{\iota_i} = t\iota_i$ for every $i \in I$.

Since t is the unique candidate for the amalgamation of the $\{t_f\}_{f \in S}$, it suffices to prove $t_f = tf \in T(Y)$ for any $f \colon Y \to X \in S$. We define Y_i by the following pullback square

$$\begin{tikzcd}
Y_i \arrow[r, "f_i"] \arrow[d, tail, "\tau_i"'] \arrow[dr, phantom, "\lrcorner", very near start] & X_i \arrow[d, tail, "\iota_i"] \\
Y \arrow[r, "f"] & X.
\end{tikzcd}$$

(Notice that the existence of those shape of pullbacks is ensured by the κ-extensivity.) Since we have $T(Y) \cong \prod_{i \in I} T(Y_i)$, it suffices to prove that $(t_f)\tau_i = (tf)\tau_i$ for every $i \in I$. This holds since $(t_f)\tau_i = t_{f\tau_i} = t_{\iota_i f_i} = t_{\iota_i} f_i = t\iota_i f_i = (tf)\tau_i$. $\square$

Corollary 2.8. *For a regular cardinal κ and a small κ-extensive category $\mathcal{C}$, the κ-extensive topology $J_{\kappa\text{-}ext}$ is subcanonical.*

Proof. This immediately follows from Theorem 2.7 ☐

The case where $\kappa = 2$ is worth mentioning:

Corollary 2.9. *A Grothendieck topos $\mathcal{E}$ is a 2-Gaeta topos if and only if it is a presheaf topos.*

Proof. For any small category $\mathcal{C}$, we obtain the category $\mathcal{C}^{\triangleleft}$ by formally adding a(n new) initial object to $\mathcal{C}$. Since the 2-extensivity is equivalent to the existence of a strict initial object, the resulting category $\mathcal{C}^{\triangleleft}$ is 2-extensive. Conversely, one can easily observe that every 2-extensive category $\mathcal{D}$ is of the form of $\mathcal{D} \simeq \mathcal{C}^{\triangleleft}$ with a small category $\mathcal{C}$. Then, the equivalence of categories

$$\mathbf{Sh}(\mathcal{C}^{\triangleleft}, J_{2\text{-ext}}) \simeq \mathbf{PSh}(\mathcal{C})$$

provided by Theorem 2.7 completes the proof. ☐

Example 2.10 (Condensed)**.**

Example 2.11 (Bornological)**.** [Law06][Law88]

Example 2.12 (Coextensive algebras)**.** [Law08][MM19]

3. From κ-extensive site to projective objects

3.1. Sites generated by free sieves.

Definition 3.1. For a small category $\mathcal{C}$, we say a sieve S on an object X is **free** if the sieve S, regarded as a presheaf on $\mathcal{C}$, is a small coproduct of representable presheaves.

In other words, a sieve S is free if and only if there is a famiy of elements $\{f_i \colon X_i \to X\}_{i\in I} \subset S$ satisfying the following equivalent conditions:

(1) Every element $f \colon Y \to X \in S$ is decomposed into

$$\begin{tikzcd} Y \arrow[rr, "g"] \arrow[dr, dashed, "\exists! h"'] & & X \\ & X_i \arrow[ur, dashed, "\exists! i\in I\ f_i"'] & \end{tikzcd}$$

with a unique pair of $i \in I$ and $h \colon Y \to X_i$.

(2) In the presheaf category $\mathbf{PSh}(\mathcal{C})$, the canonical morphism

$$\coprod_{i\in I} よ(X_i) \to S$$

is an isomorphism.

(3) For any presheaf F, the morphism

$$\mathbf{PSh}(\mathcal{C})(S, F) \to \prod_{i\in I} F(Y_i)$$

is bijective.

Remark 3.2. If $\mathcal{C}$ is Cauchy complete, a sieve (or more generally, any presheaf) S is free if and only if S is projective.

Definition 3.3. For a small category $\mathcal{C}$, we say a Grothendieck topology J is **generated by free sieves** if, for any J-covering sieve S, there exists a free J-covering subsieve $S' \subset S$.

Example 3.4. The trivial topology $(\mathcal{C}, J_{\text{triv}})$ is generated by free sieves. In fact, the maximum sieve on each object X is free.

Proposition 3.5. *For a regular cardinal κ and a κ-extensive category $\mathcal{C}$, we consider its full subcategory $\mathcal{C}'$ that has all objects in $\mathcal{C}$ except the initial object. Then the restriction of $J_{\kappa\text{-}ext}$ on $\mathcal{C}'$, which is denoted by $J'_{\kappa\text{-}ext}$, is a Grothendieck topology on $\mathcal{C}'$ generated by free sieves. Furthrermore, we have $\mathbf{Sh}(C, J_{\kappa\text{-}ext}) \simeq \mathbf{Sh}(\mathcal{C}', J'_{\kappa\text{-}ext})$.*

Proof. Since $(\mathcal{C}, J_{\kappa\text{-ext}})$ is subcanonical (Theorem 2.8) and the initial object is covered by the empty sieve, the comparison lemma memo: check and cite implies that $(\mathcal{C}', J'_{\kappa\text{-ext}})$ is a site and $\mathbf{Sh}(\mathcal{C}, J_{\kappa\text{-ext}}) \simeq \mathbf{Sh}(\mathcal{C}', J'_{\kappa\text{-ext}})$. We prove that the site $(\mathcal{C}', J'_{\kappa\text{-ext}})$ is generated by free sieves. If a sieve S on an object $X \in \mathrm{ob}(\mathcal{C}')$ is a $J'_{\kappa\text{-ext}}$-covering sieve, then S contains a κ-small coproduct diagram $\{\iota_i \colon X_i \to X\}_{i\in I} \subset S$. Let $S' \subset S$ be the subsieve generated by the family $\{\iota_i \colon X_i \rightarrowtail X\}_{i\in I}$. For any morphism $f \colon Y \to X \in S'$, f factors through a unique ι_i, since Y is not initial in $\mathcal{C}$ and κ-small coproducts are disjoint in $\mathcal{C}$. This proves that S' is a free subsieve of S. □

3.2. Site to projective objects.

Definition 3.6. We say that a site $(\mathcal{C}, J)$ **witness** that a Grothendieck topos $\mathcal{E}$ has enough projectives if we have $\mathcal{E} \simeq \mathbf{Sh}(\mathcal{C}, J)$ and the inclusion functor

$$\mathbf{Sh}(\mathcal{C}, J) \hookrightarrow \mathbf{PSh}(\mathcal{C})$$

preserves epimorphisms.

Notice that a site $(\mathcal{C}, J)$ witnesses its sheaf topos $\mathbf{Sh}(\mathcal{C}, J)$ has enough projectives if and only if, for each $x \in \mathrm{ob}(\mathcal{C})$, its represented sheaf $\mathbf{a}よ(x)$ is projective. A Grothendieck topos $\mathcal{E}$ has enough projectives objects, if and only if it admits a site $(\mathcal{C}, J)$ that witnesses that $\mathcal{E}$ has enough projectives.

Lemma 3.7 (Epimorphisms in a sheaf topos)**.** *For a site $(\mathcal{C}, J)$, a morphism of sheaves $f \colon A \to B$ is epic in the topos $\mathbf{Sh}(\mathcal{C}, J)$ if and only if, for any $X \in \mathrm{ob}(\mathcal{C})$ and $b \in B(X)$, there exists a J-covering sieve S and a (not necessarily S-matching) family $\{a_h \in A(Y)\}_{h\colon Y\to X\in S}$, such that $bh = f_y(a_h) \in B(Y)$ for any $h \colon Y \to X \in S$.*

Proof. The morphism f is an epimorphism if and only if its image coincides with B. Since the image of f in the sheaf topos is the sheafification of the objectwise image, we obtain the above description. □

Proposition 3.8. *If a site $(\mathcal{C}, J)$ is generated by free sieves, then $(\mathcal{C}, J)$ witnesses that $\mathbf{Sh}(\mathcal{C}, J)$ has enough projectives.*

Proof. Suppose that the site $(\mathcal{C}, J)$ is generated by free sieves. Let $f \colon A \to B$ be an epimorphism in the topos $\mathbf{Sh}(\mathcal{C}, J)$. Take an arbitrary object $X \in \mathrm{ob}(\mathcal{C})$ and $b \in B(X)$. We will construct $a \in A(X)$ such that $f_X(a) = b$.

Since f is epic, Theorem 3.7 provides a J-covering sieve S on the object X, and $\{a_h \in A(Y)\}_{h\colon Y\to X\in S}$, such that $bh = f_Y(a_h) \in B(Y)$ for any $h \colon Y \to X$. Using the assumption that $(\mathcal{C}, J)$ is generated by free sieves, we can take a free J-covering subsieve $S' \subset S$, and a family of elements $\{h_i \colon X_i \to X\}_{i\in I} \subset S'$ that witnesses the isomorphism

$$\coprod_{i\in I} よ(X_i) \cong S'.$$

Since A is J-sheaf and S' is free, we obtain a bijection

$$A(X) \cong \mathbf{PSh}(\mathcal{C})(よ(X), A) \cong \mathbf{PSh}(\mathcal{C})(S', A) \cong \prod_{i\in I} A(X_i)$$

and the unique element $a \in A(X)$ such that $ah_i = a_{h_i}$ for any $i \in I$.

We prove that $f_X(a) = b$. Since S' is a J-covering, it suffices to prove that $f_X(a)h_i = bh_i$ for each $i \in I$, which is verified by

$$f_X(a)h_i = f_{X_i}(ah_i) = f_{X_i}(a_{h_i}) = bh_i.$$

This completes the proof. □

Corollary 3.9. *If a Grothendieck topos $\mathcal{E}$ is a κ-Gaeta topos for a regular cardinal κ, then $\mathcal{E}$ has enough projectives.*

4. From projective objects to κ-extensive sites

4.1. **Preliminaries on projective objects.** We adopt the following terminology.

- A subobject $\iota\colon S \rightarrowtail X$ is called a **retract** if ι is a split monomorphism.
- A subobject $\iota\colon S \rightarrowtail X$ is called a **summand** if there exists another subobject $S' \rightarrowtail X$ such that

  $$S \rightarrowtail X \leftarrowtail S'$$

  is a coproduct diagram.

Lemma 4.1 (Closure properties of projective objects)**.** *For a category $\mathcal{C}$, projective objects satisfy the following closure properties.*

- *A retract of a projective object is projective.*
- *A small coproduct of projective objects is projective.*
- *If the category $\mathcal{C}$ is extensive, a summand of a projective object is projective.*

4.2. **topos with enough projectives has projective generating sets.**

Definition 4.2. A category $\mathcal{E}$ has (externally) **enough projectives,** if, for any object $X \in \mathrm{ob}(\mathcal{E})$, there exists a projective object P and an epimorphism $P \twoheadrightarrow X$.

Proposition 4.3 (Morgan Rogers [mathoverflow 316588])**.** *For a Grothendieck topos $\mathcal{E}$, the following conditions are equivalent:*

1. *$\mathcal{E}$ has enough projectives.*
2. *$\mathcal{E}$ has a (small) generating set $\mathcal{P}$ consisting of projective objects.*

Proof. We first prove (1) $\implies$ (2). Fix a generating set $\mathcal{G}$ of $\mathcal{E}$. If $\mathcal{E}$ has enough projectives for each object $g \in \mathcal{G}$, we can take a projective object P_g equipped with an epimorphism $P_g \twoheadrightarrow g$. This shows that $\mathcal{P} := \{P_g \mid g \in \mathcal{G}\}$ is a generating set consisting of projective objects.

Next, we prove (2) $\implies$ (1). For any object $X \in \mathcal{E}$, the assumption implies that there exists a jointly epimorphic small family of morphisms from projective objects $\{a_\lambda\colon P_\lambda \to X\}_{\lambda\in\Lambda}$, where $P_\lambda \in \mathcal{P}$. Theorem 4.1 implies that

$$\coprod_{\lambda\in\Lambda} P_\lambda \twoheadrightarrow X$$

is the morphism from a projective object. □

4.3. **κ-narrow objects.** To extract a κ-extensive site from a given topos with enough projectives, we consider the notion of being κ-narrow.

Definition 4.4. For a (possibly finite) cardinal κ, an object $X \in \mathrm{ob}(\mathcal{E})$ of a Grothendieck topos $\mathcal{E}$ is said to be **κ-narrow** κ if for any κ-coproduct decomposition

$$X \cong \coprod_{\alpha\in\kappa} X_\alpha$$

there exists $\alpha \in \kappa$ such that X_α is initial.

Remark 4.5. Being κ-narrow is analogous to the presentability of an object. An object X is κ-narrow if and only if $\mathcal{E}(X,-)\colon \mathcal{E} \to \mathbf{Set}$ preserves the filtered colimit diagram $\kappa = \cup_{S\subset\kappa,|S|<\kappa} S$.

Example 4.6. An object X is 2-narrow if and only if X is initial or connected.

Example 4.7. For a locally connected topos $\mathcal{E}$, an object X is κ-narrow if and only if $|\pi_0(X)| < \kappa$. In particular, a set X is κ-narrow (in the topos of sets $\mathbf{Set}$) if and only if $|X| < \kappa$.

Example 4.8. In the sheaf topos over the Cantor space $\mathbf{Sh}(2^{\mathbb{N}})$, the terminal object is $\aleph_0$-narrow.

Proposition 4.9 (Closure properties of κ-narrow objects)**.** *For a Grothendieck topos $\mathcal{E}$ and a regular cardinal κ,*

- *A κ-small coproduct of κ-narrow objects is κ-narrow.*
- *A summand of a κ-narrow object is κ-narrow.*

Proof. Let X be a κ-small coproduct of κ-narrow objects $\{X_\lambda\}_{\lambda\in\Lambda}$ ($|\Lambda| < \kappa$).

$$X = \coprod_{\lambda\in\Lambda} X_\lambda$$

Take an arbitrary κ-coproduct decomposition $X = \coprod_{\alpha\in\kappa} Y_\alpha$. The infiniary extensivity of $\mathcal{E}$ implies that

$$X = \coprod_{\lambda\in\Lambda} X_\lambda \cong \coprod_{\lambda\in\Lambda}\coprod_{\alpha\in\kappa} X_\lambda \times_X Y_\alpha.$$

For each $\lambda \in \Lambda$, we define $I_\lambda \coloneqq \{\alpha \in \kappa \mid X_\lambda \times_X Y_\alpha \text{ is not initial.}\}$ Since each X_λ is κ-narrow, we have $|I_\lambda| < \kappa$. The regularity of κ implies that $\bigcup_{\lambda\in\Lambda} I_\lambda \subsetneq \kappa$. For an element $\alpha \in \kappa \setminus \left(\bigcup_{\lambda\in\Lambda} I_\lambda\right)$, we have

$$Y_\alpha \cong \coprod_{\lambda\in\Lambda} X_\lambda \times_X Y_\alpha \cong \coprod_{\lambda\in\Lambda} \emptyset \cong \emptyset.$$

This completes the proof of the former statement.

The latter statement is easier to prove. ☐

The following lemma is easy, but essential.

Lemma 4.10. *For a regular cardinal κ, a Grothendieck topos $\mathcal{E}$, a κ-narrow projective object P, and a small (not necessarily κ-small) family of morphisms $\{f_i \colon X_i \to P\}_{i\in I}$, the following conditions are equivalent:*

- *$\{f_i \colon X_i \to P\}_{i\in I}$ is jointly epimorphic.*
- *There exists a κ-small corpoduct decomposition $\coprod_{\lambda\in\Lambda} P_\lambda$ ($|\Lambda| < \kappa$) such that every inclusion $P_\lambda \rightarrowtail P$ factors through some $f_i \colon X_i \to P$.*

Proof. It is easy to prove that the latter condition implies the former. We prove the opposite.

Assume that $\{f_i \colon X_i \to P\}_{i\in I}$ is jointly epimorphic. Then we obtain the canonical epimorphism $\sum_{i\in I} f_i \colon \coprod_{i\in I} X_i \twoheadrightarrow P$. The projectivity of P ensures the existence of a section $\coprod_{i\in I} X_i \leftarrowtail P \colon s$. Since $\mathcal{E}$ is infinitary extensive, the morphism s induces the I-coproduct decomposition $P \cong \coprod_{i\in I} P_i$ by the pullback diagram

$$\begin{tikzcd}
X_i \arrow[d, tail] & P_i \arrow[l, tail] \arrow[d, tail] \arrow[dl, phantom, "\lrcorner"] \\
\coprod_{i\in I} X_i \arrow[r, bend right, two heads, "\sum_{i\in I} f_i"'] & P. \arrow[l, tail, "s"']
\end{tikzcd}$$

This implies that each inclusion $P_i \rightarrowtail P$ factors through $f_i \colon X_i \to P$. Since P is κ-narrow, the subset $\Lambda \coloneqq \{i \in I \mid P_i \ncong \emptyset\}$ is κ-small, and we have a κ-small coproduct decomposition $P \cong \coprod_{\lambda\in\Lambda} P_\lambda$. This completes the proof. ☐

4.4. The κ-extensive site of κ-narrow projective objects.

Definition 4.11. For a Grothendieck topos $\mathcal{E}$ and a regular cardinal κ, the full subcategory of κ-narrow projective objects is denoted by $\mathcal{C}_\kappa$.

Example 4.12. For a small category $\mathcal{C}$, the category $\mathcal{C}_2 \subset \mathbf{PSh}(\mathcal{C})$ for $\kappa = 2$ is equivalent to $\overline{\mathcal{C}}^{\triangleleft}$, where $\overline{\mathcal{C}}$ denotes tha Cauchy completion of $\mathcal{C}$. This is the opposite construction of Theorem 2.9.

Theorem 4.13. *For a Grothendieck topos $\mathcal{E}$ and a regular cardinal κ,*

- *the full subcategory $\mathcal{C}_\kappa \subset \mathcal{E}$ is essentially small and κ-extensive.*
- *the canonical topology J_{can} on $\mathcal{C}_\kappa$ coincides with the κ-extensive topology $J_{\kappa\text{-}ext}$.*

Furthermore, there exists a regular cardinal κ such that $\mathcal{C}_\kappa$ is a generating full subcategory of $\mathcal{E}$, if and only if the topos $\mathcal{E}$ has enough projectives.

Proof. First, we prove the essential smallness. Fix a generating set G of the topos $\mathcal{E}$. Theorem 4.10 implies that any object in $\mathcal{C}_\kappa$ is a κ-small coproduct of subobjects of objects in G. This implies that there are at most small number of isomorphism classes in $\mathcal{C}_\kappa$.

Theorem 4.1 and Theorem 4.9 implies that $\mathcal{C}_\kappa \hookrightarrow \mathcal{E}$ is closed under taking summands and κ-small coproducts. This implies that $\mathcal{C}_\kappa$ inherits the κ-extensivity of $\mathcal{E}$.

Theorem 4.10 implies that a sieve $S \subset \mathcal{C}_\kappa(-, P)$ belongs to J_{can} if and only if it contains a κ-small coproduct diagram.

The last statement follows from the fact that every object X in a Grothendieckt topos is κ-narrow for sufficiently large cardinal κ. memo: Well-foundedness □

Corollary 4.14. *If a Grothendieck topos $\mathcal{E}$ has enough projectives, we have*

$$\mathcal{E} \simeq \mathbf{Sh}(\mathcal{C}_\kappa, J_{\kappa\text{-}ext})$$

for sufficiently large regular cardinal κ, where $\mathcal{C}_\kappa$ is the essentially small κ-extensive full subcategory consisting of κ-narrow projective objects, and $J_{\kappa\text{-}ext}$ is its κ-extensive topology.

5. Conclusion

Theorem 5.1 (Main Theorem)**.** *For a Grothendieck topos $\mathcal{E}$, the following conditions are equivalent:*

1. *$\mathcal{E}$ has enough projective objects.*
2. *There is a regular cardinal κ such that $\mathcal{E}$ is a κ-Gaeta topos (Theorem 2.6).*
3. *$\mathcal{E}$ is equivalent to a sheaf topos $E \simeq \mathbf{Sh}(\mathcal{C}, J)$, where $(\mathcal{C}, J)$ is a small site generated by free sieves.*

Proof. Theorem 4.14 proves the implication (1) $\implies$ (2). Theorem 3.5 proves the implication (2) $\implies$ (3). Theorem 3.8 proves the implication (3) $\implies$ (1). □

Example 5.2 (Presheaves)**.** A presheaf topos $\mathbf{PSh}(\mathcal{C})$ has enough projective, since the trivial topology on $\mathcal{C}$ is generated by free sieves.

Example 5.3 (Condensed sets)**.** For a strong limit cardinal λ, the topos of λ-condensed sets has enough projectives, since the topos is equivalent to the sheaf topos over the $\aleph_0$-extensive site of λ-extremely disconnected spaces.

Example 5.4. memo: This may subsume [Dup89]

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