DIFFERENTIAL CALCULUS OF GAMES

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Abstract.

Contents

Idea 0.1. Our goal is to clarify the implicit differential structure behind combinatorial game theory and demystify the Nim-sum.

1. Differential operator on games

1.1. Categorical differential operator.

Definition 1.1 (Ryuya Hora)**.** A **pointed game** is a pair $(\mathbb{X}, x)$ of a game $\mathbb{X} = (X, \to)$ (in the sense of [Hor]) and an element $x \in X$. The category of pointed games and point-preserving game morphisms is denoted by $\mathbf{Games}_*$.

Definition 1.2 (Ryuya Hora)**.** Let $\mathbf{Fam}(\mathbf{Games}_*)$ denote the free coproduct cocompletion of $\mathbf{Games}_*$.

An object of $\mathbf{Fam}(\mathbf{Games}_*)$ is an indexed family of pointed games $\{(\mathbb{X}_\lambda, x_\lambda)\}_{\lambda\in\Lambda}$.

Proposition 1.3 (Ryuya Hora)**.** *The conway addition $\otimes$ on* **Games** *is extended to a monoidal structure on* $\mathbf{Fam}(\mathbf{Games}_*)$ *by*

$$\{(\mathbb{X}_i, x_i)\}_{i\in I} \otimes \{(\mathbb{Y}_j, y_j)\}_{j\in J} \coloneqq \{(\mathbb{X}_i \otimes \mathbb{Y}_j, x_i \otimes y_j)\}_{(i,j)\in I\times J}.$$

Furthermore, it defines a rig structure on $\mathbf{Fam}(\mathbf{Games}_*)$ *with the categorical coproducts.*

- $0 = \{\}$ *(empty family)*
- $1 = \{(\{x\}, x)\}_{*\in\{*\}}$

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- $X \otimes (Y + Z) = X \otimes Y + X \otimes Z$

Definition 1.4 (Ryuya Hora)**.** The **differential operator**

$$\partial \colon \mathbf{Games}_* \to \mathbf{Fam}(\mathbf{Games}_*)$$

is defined by

$$\partial(\mathbb{X}, x) := \{(\mathbb{X}, x')\}_{x \to x'}.$$

The unique coproduct-preserving extension

$$\partial \colon \mathbf{Fam}(\mathbf{Games}_*) \to \mathbf{Fam}(\mathbf{Games}_*)$$

is also denoted by the same symbol ∂.

Proposition 1.5 (Ryuya Hora)**.** *The differential operator ∂ satisfies the categorified Leibniz rule:*

$$\partial(X \otimes Y) \cong \partial X \otimes Y + X \otimes \partial Y$$

This is quite similar to the situation of [Joy81, Proposition 5]. hora: This should be a **differential 2-rig** [LT21]

By microcosm principle, these differential structures on the category $\mathbf{Fam}(\mathbf{Games}_*)$ will be reflected on its terminal object $\{(\mathbb{H}, h)\}_{h \in \mathbb{H}}$.

Question 1.6. Is there a "composition" of a game such that the categorified chain rule holds?

Question 1.7. Is $\mathbf{Fam}(\mathbf{Games}_*)$ a Grothendieck topos? hora: $\mathbf{Games}_*$ shares a lot of categorical structures with $\mathbf{Set}_*$, and $\mathbf{Fam}(\mathbf{Set}_*)$ is equivalent to the topos of idempotents.

1.2. Differential rig and differential invariant of games.

Definition 1.8. A **differential rig** is a (possibly non-commutative) rig (= semiring) $(A, 0, 1, +, \times)$ equipped with a unary operation $\partial \colon A \to A$ such that

- $\partial(0) = 0$
- $\partial(a + b) = \partial a + \partial b$
- $\partial(1) = 0$
- $\partial(ab) = (\partial a)b + a(\partial b)$

This is equivalent to assuming

$$\partial\left(\sum_{i=1}^{n} a_i\right) = \sum_{i=1}^{n} \partial a_i$$

and

$$\partial\left(a_1 \times \cdots \times a_n\right) = \sum_{i=1}^{n} \left(a_1 \times \cdots \times a_{i-1} \times (\partial a_i) \times a_{i+1} \times \cdots \times \cdots \times a_n\right)$$

for every non-negative integer n.

Definition 1.9. A **differential invariant** of $\mathbf{Fam}(\mathbf{Games}_*)$, is a function[1] $p \colon \mathrm{ob}(\mathbf{Fam}(\mathbf{Games}_*)) \to A$ to a differential rig A, that respects $0, 1, +, \times, \partial$.

Example 1.10 (Differential rig of polynomial)**.** The commutative ring of polynomials $\mathbb{Z}[x]$ equipped with the usual notion of differentiation $\partial \colon \mathbb{Z}[x] \to \mathbb{Z}[x]$ is a differential rig. Theorem 1.29 is an example of a differential invariant of games.

[1] which is a proper-class large function

Example 1.11 (max-plus algebra)**.** The max-plus algebra $\mathbb{N} \cup \{-\infty\}$ admits a unique differential rig structure such that $\deg\colon \mathbb{N}[x] \to \mathbb{N} \cup \{-\infty\}$ is a differential rig homomorphism. More concretely,

$$\partial n := \begin{cases} n-1 & (n > 0) \\ -\infty & (n \leq 0) \end{cases}$$

Example 1.12 (Trivial differential structure)**.** For any rig A, $\partial(a) := 0$ defines a trivial differential structure regarding every element constant.

Definition 1.13. Let $\Omega = \{\bot, \top\}$ denote the rig of truth values. hora: This is sometimes regarded as "$\mathbb{F}_1$" by some authors, like Lawvere or Connes. [Law08; CC17] Why...? Here might be some relationship with rig-(tropical-)geometry [Men21]. See also theorem 1.21 The addition is $\vee$ and the multiplication is $\wedge$. For a monoid M, let $\Omega[M]$ denote the free Ω-rig generated by the monoid M, i.e., the left adjoint for the forgetful functor

$$\begin{tikzcd} \Omega/\mathbf{Rig} \arrow[r, "U"'] & \mathbf{Rig} \arrow[r, "U"'] & \mathbf{Monoid}. \arrow[ll, bend right, "\Omega[-]"'] \end{tikzcd}$$

Lemma 1.14. *For a monoid $(M, e, *)$, the resulting rig $\Omega[M]$ is isomorphic to $\mathcal{P}_{\mathrm{fin}}(M)$ equipped with*

zero: $0 := \emptyset$,
one: $1 := \{e\}$,
addition: $A + B := A \cup B$, *and*
multiplication: $A \times B := \{a * b \mid a \in A, b \in B\}$

Theorem 1.15. *For a monoid M, the following data are in one-to-one correspondence:*

- *Differential rig structure on $\Omega[M]$.*
- *Monoid object in the monoidal category $(\mathbf{Coalg}_{\mathcal{P}_{\mathrm{fin}}}, \otimes^{2})$ whose underlying monoid is M.*

Proof. A differential rig structure

$$\partial\colon \Omega[M] \to \Omega[M]$$

is nothing other than a function (= $\mathcal{P}_{\mathrm{fin}}$-coalgebra structure)

$$\delta\colon M \to \Omega[M] \cong \mathcal{P}_{\mathrm{fin}}(M)$$

satisfying

(1) $\delta(e) = \emptyset$ and
(2) $\delta(ab) = \delta(a)\{b\} \cup \{a\}\delta(b) = \{a'b \mid a' \in \delta(a)\} \cup \{ab' \mid b' \in \delta(b)\}$.

The first condition says that $1 \to M$ is a $\mathcal{P}_{\mathrm{fin}}$-coalgebra morphis, and the second says that $M \times M \to M$ is a $\mathcal{P}_{\mathrm{fin}}$-coalgebra morphism. $\square$

hora: this can be generalized to differential rig, like $\mathbb{Z}[x]$ or $\Omega[\mathbb{H}]$. The first provides the generating polynomial above, and the latter might provide Ryo Suzuki's pov. The commutative ring $\mathbb{Z}[\mathbb{H}]$ does not satisfy the Leibniz rule, but the ordered semiring $\mathbb{N}[\mathbb{H}]$ satisfies the lax version of the Leibniz rule.

If we want to consider $(\mathbf{Games}, \otimes)$, (not $(\mathbf{Coalg}_{\mathcal{P}_{\mathrm{fin}}}, \otimes)$), we need the notion of "finite degree".

Definition 1.16 (Finite degree differential rig)**.** For a differential rig A, we adopt the following terminology.

- The **degree** of an element $a \in A$ is the minimum natural number $n \in \mathbb{N}$ such that $\delta^n(a) = 0$, if such n exists.
- An element is said to be **of finite degree** if it has a degree (in $\mathbb{N}$).

[2] conway addition

- A differential rig A is said to be **of finite defree**, if every element is of finite degree.

Example 1.17. For the differential rig $C^\infty(\mathbb{R})$, an element $f\colon \mathbb{R} \to \mathbb{R}$ is of finite degree if and only if f is a polynomial. While $C^\infty(\mathbb{R})$ is not of finite degree as a differential rig, $\mathbb{Z}[x]$ is a typical example of a differential rig of finite degree.

Theorem 1.18. *For a monoid M, there is a one-to-one correspondence between*

Game enrichment: *A monoid object in* $(\mathbf{Games}, \otimes)$ *with the underlying monoid* M.
Differential str: *Differential rig structure of finite degree on* $\Omega[M]$

Corollary 1.19. *Since the terminal monoid* $\mathbb{H}$ *in* $(\mathbf{Games}, \otimes)$ *is obviously a monoid object, we obtain a differential rig* $\Omega[\mathbb{H}]$ *of finite degree. Via the bijection* $\mathbb{H} = \mathcal{P}_{\mathrm{fin}}(\mathbb{H}) \cong \Omega[\mathbb{H}]$, *this structure is written down as:*

zero: $0 \coloneqq \{\} = \emptyset$
sum: $A + B \coloneqq A \cup B$
one: $1 = \{\{\}\}$
prod: $AB \coloneqq \{a * b \mid a \in A, b \in B\}$, *where* $a * b$ *is the terminal monoid structure* ($a * b \coloneqq \{a' * b \mid a' \in a\} \cup \{a * b' \mid b' \in b\}$).
diff: $\partial A \coloneqq \bigcup_{a \in A} a = \{a_2 \mid a_2 \in a_1 \in A\}$.

Remark 1.20. This differential rig structure shares the same idea with theorem 1.5. It is better to consider the differential rig $\mathbb{H}$ as an algebra of **lists**[3] **of games** rather than an algebra of games. Each element of the rig is regarded as a finite list of (pointed impartial) games. The addition is a concatenation of the list of games, the multiplication is the game-wise Conway addition, and the differential is the list of all possible next states.

Remark 1.21 (Max-plus algebra in the differential rig of games)**.** The max-plus algebra $\mathbb{N} \cup \{-\infty\}$ (theorem 1.11) is embedded into the rig of game families $\mathbb{H}$ respecting the differential structure! First, we define $[n] \in \mathbb{H}$ by $[0] = \emptyset$ and $[n+1] = \{[n]\}$. We can recursively prove that $[n] * [m] = [n+m]$. A function $\iota\colon \mathbb{N} \cup \{-\infty\} \to \mathbb{H}$ defined by

$$\iota(n) \coloneqq \begin{cases} \{[0], \ldots, [n]\} & (n \geq 0) \\ \{\} & (n = -\infty) \end{cases}$$

is an embedding of differential rig.

hora: There should be a similar differential rig for Partizan games.
hora: We need to consider a lax morphism between differential rigs

1.3. **Theory of Rota-Baxter rig.** In this subsection, we aim to provide a way to decategorify the differential structure of $\mathbf{Fam}(\mathbf{Games}_*)$, defining a differential invariant theorem 1.9. More concretely, we will define the notion of **Rota-Baxter rig** (theorem 1.22), and prove the followings:

(1) Game families $\mathbb{H} \cong \mathcal{P}_{\mathrm{fin}}(\mathbb{H})$ form a Rota-Baxter rig. (theorem 1.25)
(2) Rota-Baxter rig A provides a differential invariant.

Definition 1.22. A **Rota-Baxter rig** is a differential rig A, equipped with an **integral operator** $\int$, that is, a function $\int\colon A \to A$ satisfying

Fundamental theorem of calculus: $\partial \int f = f$.
Nullary Rota-Baxter equation: $1 = \int 0$
Binary Rota-Baxter equation: $(\int f)(\int g) = \int (f(\int g) + (\int f) g)$

[3] it is more accurate to say families

Remark 1.23 (n-ary Rota-Baxter equations)**.** In the definition, we assume two Rota-Baxter equations, which can be unified to **n-ary Rota-Baxter equation**

$$\left(\int f_1\right) \times \cdots \times \left(\int f_n\right) = \int \left(\sum_{i=1}^{n} \left(\int f_1\right) \times \cdots \times \left(\int f_{i-1}\right) \times f_i \times \left(\int f_{i+1}\right) \times \cdots \times \left(\int f_n\right)\right).$$

Assuming the nullary and binary Rota-Baxter equations is equivalent to assuming all n-ary equations for $n \in \mathbb{N}$. This is an example of [Biased definition].

Example 1.24. For any $\lambda \in \mathbb{R}$, the function

$$\int : f(x) \mapsto \lambda + \int_0^x f(t)dt$$

satisfies the first condition $\partial \int f = f$. This satisfies the (n-ary) Rota-Baxter equations if and only if $\lambda = 1$.

Theorem 1.25 (Rota-Baxter rig of game families (Ryo Suzuki))**.** *The differential rig $\mathbb{H}$ (theorem 1.19) equipped with the integral operator*

$$\int A \coloneqq \{A\}$$

is a Rota-Baxter rig.

Proof. The first two equalities are easily proven as

$$\partial \int A = \{a_2 \mid a_2 \in a_1 \in \{A\}\} = A,$$

and

$$\int 0 = \{0\} = 1.$$

For the last equality, we have

$$\begin{aligned}
\int A \times \int B &= \{A * B\} \\
&= \{\{a * B \mid a \in A\} \cup \{A * b \mid b \in B\}\} \\
&= \{A \times \{B\} \cup \{A\} \times B\} \\
&= \int \left(A \times \int B + \int A \times B\right)
\end{aligned}$$

This completes the proof. ∎

Analysis	Game family
0	$\{\}$
1	$\{\{\}\}$
$+$	$\cup$
$\times$	$\{a * b \mid a \in A, b \in B\}$
∂	$\partial A = \bigcup_{a \in A} a$
$\int$	$\int A = \{A\}$

TABLE 1. List of the correspondence

Lastly, we will prove that a Rota-Baxter rig provides a differential invariant.

Theorem 1.26. *For a Rota-Baxter algebra A, and a game $\mathbb{X}$, we recursively define a function $F_{(\mathbb{X},-)}\colon X \to A$ by*

$$F_{(\mathbb{X},x)} \coloneqq \int \left(\sum_{x \to x'} F_{(\mathbb{X},x')} \right).$$

For a finite family of games $(\mathbb{X}_i, x_i)_{i \in I}$, where $|I| < \infty$, we define

$$F_{(\mathbb{X}_i,x_i)_{i\in I}} \coloneqq \sum_{i \in I} F_{(\mathbb{X}_i,x_i)}.$$

These data define a differential invariant of games (theorem 1.9), i.e., we have

zero: $F_0 = 0$,
one: $F_1 = 1$,
addition: $F_{\mathcal{X}+\mathcal{Y}} = F_{\mathcal{X}} F_{\mathcal{Y}}$,
multiplication: $F_{\mathcal{X}+\mathcal{Y}} = F_{\mathcal{X}} F_{\mathcal{Y}}$, *and*
Differential: $F_{\partial \mathcal{X}} = \partial F_{\mathcal{X}}$.

We need a little preparation to prove this theorem. In a Rota-Baxter rig, $\int \partial a$ is not generally equal to a. We will give a name for such elements:

Definition 1.27. For a Rota-Baxter rig $(A, 0, 1, +, \times, \partial, \int)$, an element $a \in A$ is said to be **normal**, if it satisfies the following equivalent conditions:

- $\int \partial a = a$.
- There is an element $b \in A$ such that $a = \int b$.

Lemma 1.28. *In a Rota-Baxter rig A, every finite product of normal elements is again normal, i.e.,*

- 1 *is normal.*
- *If a and b are normal, then ab is normal.*

Proof. This immediately follows from the n-ary Rota-Baxter equation. □

Proof of theorem 1.26. The construction in theorem 1.26 preserves zero and binary addition, by definition. It preserves the derivative, due to the

$$\partial F_{(\mathbb{X},x)} = \partial \int \sum_{x \to x'} F_{(\mathbb{X},x')} = \sum_{x \to x'} F_{(\mathbb{X},x')} = F_{\partial(\mathbb{X},x)}$$

and the lineality of ∂. It preserves 1, by the nullary Rota-Baxter equation:

$$F_1 = F_{(1,*)} = \int \sum_{* \to x'} F_{(1,x')} = \int 0 = 1.$$

Lastly, we prove that it preserves binary products. Notice that $F_{(\mathbb{X},x)}$ is always normal for every game $(\mathbb{X},x)$, while $F_{(\mathbb{X}_i,x_i)_{i\in I}}$ is not normal in general. We prove that it preserves binary products by induction:

$$\begin{aligned}
F_{(\mathbb{X},x)\otimes(\mathbb{Y},y)} &= \int \left(\sum_{x\to x'} F_{(\mathbb{X},x')\otimes(\mathbb{Y},y)} + \sum_{y\to y'} F_{(\mathbb{X},x)\otimes(\mathbb{Y},y')} \right) \\
&= \int \left(\left(\sum_{x\to x'} F_{(\mathbb{X},x')} \right) \times F_{(\mathbb{Y},y)} + F_{(\mathbb{X},x)} \times \left(\sum_{y\to y'} F_{(\mathbb{Y},y')} \right) \right) \\
&= \int \left(\partial F_{(\mathbb{X},x)} \times F_{(\mathbb{Y},y)} + F_{(\mathbb{X},x)} \times \partial F_{(\mathbb{Y},y)} \right) \\
&= \int \partial \left(F_{(\mathbb{X},x)} \times F_{(\mathbb{Y},y)} \right) \\
&= F_{(\mathbb{X},x)} \times F_{(\mathbb{Y},y)}
\end{aligned}$$

The last equation follows since $F_{(\mathbb{X},x)} \times F_{(\mathbb{Y},y)}$ is normal due to theorem 1.28. □

hora: Relationship with Bouton system

hora: We can categorify the integral operator. Does it satisfy the categorified Rota-Baxter equation? Possibly No. (Count the number of elements

1.4. Generating Polynomial.

Definition 1.29 (A generating function)**.** hora: modified Here, we can define a notion of **real-valued generating function**. For an object $(\mathbb{X},x)$ in $\mathbf{Fam}(\mathbf{Games}_*)$, let $g_{(\mathbb{X},x)}(z)$ be the polynomial

$$g_{(\mathbb{X},x)}(z) = \sum_{n=0}^{\infty} p_n \frac{z^n}{n!},$$

where p_n denotes the number of "plays of length n"

$$p_n \coloneqq \{(x_1,\ldots,x_n) \in X^n \mid x \to x_1 \to \cdots \to x_n\}.$$

Proposition 1.30. *hora: modified We can extend this to a finite family of games in* $\mathbf{Fam}(\mathbf{Games}_*)$*, and we can prove that generating functions respect sum, multiplication, and derivative of games.*

Proof. This is a special case of theorem 1.26 for differential rig $C^\infty(\mathbb{R})$ with

$$\int : f(x) \mapsto 1 + \int_0^x f(t)dt$$

□

Remark 1.31 (This is not informative)**.** hora: To be modified This is not very useful for studying game-winning strategy since N, P-games might share the generating function. For example, there are two games with the generating function hora: wrong $\frac{x^3}{6} + x^2$, one of which is N-game and the other is P-game.

Remark 1.32. hora: To be modified Even for a non-pointed game $\mathbb{X}$, we can consider the generating function. Let R be the right adjoint to the forgetful functor

$$U \colon \mathbf{Fam}(\mathbf{Games}_*) \to \mathbf{Games} \colon (\mathbb{X}_i, x_i)_{i\in I} \mapsto \coprod_{i\in I} \mathbb{X}_i.$$

More explicitly, it is calculated by $R\mathbb{X} = (\mathbb{X}, x)_{x\in X}$. We define $g_{\mathbb{X}}$ to be the (possibly non-convergent) generating function $g_{R\mathbb{X}}$. For example, let $S \in \mathcal{P}_{\mathrm{fin}}(\mathbb{N}\setminus\{0\})$ and $\mathbb{X}_S$ be the subtraction Nim (i.e., $n \to m$ iff $n - m \in S$). Its generating function is $e^{|S|z}$.

hora: similar, but essentially different function described [here] has the data of N, P.

1.5. **Towrds the maximally informative invariant.** Both $C^\infty(\mathbb{R})$ and $\mathbb{H}$ provide a Rota-Baxter rig and, therefore, induce a differential invariant of games. $C^\infty(\mathbb{R})$ has enumerative information, and $\mathbb{H}$ has game-theoretic information (like P/N states). However, there is the maximally informative Rota-Baxter rig: **initial Rota-Baxter rig**.

Question 1.33. What is the initial Rota-Baxter rig?

2. Bouton System

suzuki: To understand the Grundy number, it might be worth considering some monoidal structure on $\mathbf{Alg}_{\mathcal{P}_{\mathrm{fin}}}$. (or Bouton monoid can be used to understand?) It seems to be worth considering that "Bouton system", which is defined below.

Definition 2.1 (Ryo Suzuki)**.** A tuple $(A, \mu\colon \mathcal{P}_{\mathrm{fin}}A \to A, e \in A, *\colon A\times A \to A$ is called **Bouton system** when

1. (A, μ) is a $\mathcal{P}_{\mathrm{fin}}$-algebra,
2. $(A, e, *)$ is a commutative monoid,
3. $\mu(\varnothing) = e$, and
4. For every $S, T \in \mathcal{P}_{\mathrm{fin}}A$, $\mu(\{\mu(S) * t, s * \mu(t) \mid s \in S, t \in T\}) = \mu(S) * \mu(T)$.

For example, the terminal game and $(\mathbb{N}, \mathrm{mex})$ can be regarded as a Bouton system. On the other hand, although P, N has a $\mathcal{P}_{\mathrm{fin}}$-algebra structure and a commutative monoid structure, it is not a Bouton system. suzuki: Can Bouton systems be regarded as monoid objects of $\mathbf{Alg}_{\mathcal{P}_{\mathrm{fin}}}$ for certain monoidal structure?

Remark 2.2 (Bouton system in terms of Rota-Baxter rig (without differentiation))**.** For any monoid M, we obtain a rig $\Omega[M] \cong \mathcal{P}_{\mathrm{fin}}(M)$. For a general rig, an **integral operator** $\int\colon A \to A$ is a function that satisfies

Nullary Rota-Baxter equation: $1 = \int 0$
Binary Rota-Baxter equation: $(\int f)(\int g) = \int (f(\int g) + (\int f)g)$

(See also theorem 1.22). There is a one-to-one correspondence between

- Bouton system structure on a monoid M, and
- An integral operator $\int\colon \Omega[M] \to \Omega[M]$ that factor through $\{\cdot\}\colon M \hookrightarrow \mathcal{P}_{\mathrm{fin}}(M)\colon m \mapsto \{m\}$,

via the composition

$$\begin{tikzcd} \mathcal{P}_{\mathrm{fin}}(M) \arrow[r, "\mu"] \arrow[rr, bend right, "\int"'] & M \arrow[r, "\{\cdot\}"] & \mathcal{P}_{\mathrm{fin}}(M). \end{tikzcd}$$

Remark 2.3 (Monoid structure w.r.t. the categorical product)**.** A monoid structure $*\colon A\times A \to A$ on a $\mathcal{P}_{\mathrm{fin}}$-algebra $\mu\colon \mathcal{P}_{\mathrm{fin}}(A) \to A$ is a monoid in the cartesian category $\mathbf{Alg}_{\mathcal{P}_{\mathrm{fin}}}$, if and only if

$$\mu(x_i)_{i\in I} * \mu(y_i)_{i\in I} = \mu(x_i * y_i)_{i\in I}$$

holds for any finite set I and a function $I \to A^2$. This is NOT Bouton system. For example, $(\mathbb{N}, \mathrm{mex})$ does not satisfy this equation. (If I is not empty and $x_i = y_i$, then the left-hand side is 0, but the right-hand side is 1.)

Proposition 2.4 (Ryo Suzuki)**.** *Let A be a $\mathcal{P}_{\mathrm{fin}}$-algebra. Then the Bouton monoid $\mathcal{B}_{\otimes,A}$ has a natural $\mathcal{P}_{\mathrm{fin}}$-algebra structure, and become a Bouton system.*

Proof. Let $B = \mathcal{B}_{\otimes,A}$. To induce a $\mathcal{P}_{\mathrm{fin}}$-algebra structure on B, it suffice to show that, if $S, T \in \mathcal{P}_{\mathrm{fin}}(\mathbb{H})$ has a same image in B, then $S, T \in \mathbb{H}$ has also a same image in B. This is verified inductively as follows: For any $U \in \mathbb{H}$,

$$S * U = \{s * U, S * u \mid s \in S, u \in U\} \tag{1}$$
$$= \{t * U, T * u \mid t \in T, u \in U\} \tag{2}$$
$$= T * U. \tag{3}$$

It immediately follows B is a Bouton system. □

suzuki: I think this proposition can be generalized as follows: Let X be a Bouton system ,A a $\mathcal{P}_{\mathrm{fin}}$-algebra, and $X \to A$ be a $\mathcal{P}_{\mathrm{fin}}$-algebra map. Then the minimal quotient monoid of X of $X \to A$ has a natural Bouton system structure. I also believe for any $\mathcal{P}_{\mathrm{fin}}$-algebra there exists a surjection from some Bouton system. I think these can be used to construct the right adjoint R of the forgetful functor F from the category of Bouton system to $\mathbf{Alg}_{\mathcal{P}_{\mathrm{fin}}}$. suzuki: Since F commute with limits, there probably exists the left adjoint L of the forgetful functor. These could be used to calculate the underlying set of (conjectual) $R(A)$.

3. Another presentation of the category of games

Let C be a category and let $T\colon C \to C$ be a monad. Let $(X, X \to TX)$ be a T-coalgebra. Then we have a map $\partial = \mu_X \circ Tf : TX \to TX$. This is an endomorphism of the T-algebra $(TX, \mu_X : T^2X \to TX)$, here we regard T as an endofunctor.

$$\begin{tikzcd}
TTX \arrow[r, "T\theta"] \arrow[d, "\mu_X"] & TTTX \arrow[r, "T\mu_X"] \arrow[d, "\mu_{TX}"] & TTX \arrow[d, "\mu"] \\
TX \arrow[r, "\theta"] & TTX \arrow[r, "\mu_X"] & TX
\end{tikzcd}$$

Conversely, if we have a endomophism ∂ of the T-algebra $(TX, \mu : T^2X \to TX)$, then we have a coalgebra $(X, \partial \circ \eta_X : X \to TX)$. This correspondence is one-to-one. suzuki: Does this reinterpretation help to understand the tensor product of games? hora: Since (TX, μ_X) is the free T-algebra, we have $\mathbf{Alg}_T(TX, TX) \cong \mathcal{C}(X, TX)$. I did not notice this!

hora: Can we rephrase 'being games', (not only 'being coalgebras')?

4. Related works

1901: Bouton proved the famous winning strategy of Nim with **Nim-sum** [Bou01].
1935, 1939: Sprague and Grundy independently find what's now called the Sprague-Grundy theorem. [Spr35] [Gru39]
1977: Joyal defined the compact closed category of combinatorial games. [Joy77]
1981: Joyal founds the category of species is the categorification of "exponential generating functions" [Joy81]
1992: founds the game semantics. [Bla92]
2006: Blute, Cockett, and Seely defined differential categories. [BCS06]
2009: **Rota-Baxter categories** are defined in [CR09].
2013: As far as I know, the first paper that deals with "differential category of games" is [LMM13].
2021: Differential 2-rigs are defined [LT21]
2025: Hora will write a paper on a new category of games.

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