THE A_2 QUIVER OVER A FINITE FIELD AND WYTHOFF MOVES

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Abstract. We classify representations of the quiver $1 \to 2$ over a finite field $\mathbb{F}_q$ as linear maps. The multiplicities of its three indecomposable representations define a map to dimension vectors. The existential quotient of single-coordinate deletion along this map is exactly the three-direction move relation of Wythoff Nim. We also express the same relation as the dimension-vector shadow of the Boolean Hall product. This is a representation-theoretic description of legal moves; it gives neither a formula nor a faster algorithm for Grundy values.

1. Two directed graphs

We write $\mathbb{N} = \{0, 1, 2, \ldots\}$.

Definition 1.1 (Wythoff graph)**.** The vertices of the Wythoff graph are the elements of $\mathbb{N}^2$. There is an edge from (a, b) to (a', b') if, for some integer $r \geq 1$,

$$(a, b) - (a', b') \in \{r(1, 0),\ r(0, 1),\ r(1, 1)\}.$$

This is the legal-move relation in Wythoff's original paper [1], written as a directed graph.

Let Q be the quiver $1 \to 2$, and let $\mathrm{Rep}_{\mathbb{F}_q}(Q)$ be its category of finite-dimensional representations. An object is a linear map of finite-dimensional $\mathbb{F}_q$-vector spaces

$$M = (V_1 \xrightarrow{f} V_2).$$

Its dimension vector is $\mathbf{dim} M = (\dim V_1, \dim V_2)$. We use the three representations

$$S_1 = (\mathbb{F}_q \to 0), \qquad S_2 = (0 \to \mathbb{F}_q), \qquad E = (\mathbb{F}_q \xrightarrow{1} \mathbb{F}_q).$$

General introductions to quiver representations include Schiffler [2] and Kirillov [3]. The following A_2 classification is their smallest example; we include the proof for completeness.

Proposition 1.2 (Classification of A_2 representations)**.** *For every representation $M = (V_1 \xrightarrow{f} V_2)$,*

$$M \cong S_1^{\oplus u} \oplus E^{\oplus k} \oplus S_2^{\oplus v}, \qquad (u, k, v) = (\dim\ker f, \operatorname{rank} f, \dim\operatorname{coker} f).$$

In particular, the three multiplicities are unique.

Proof. Choose a decomposition $V_1 = \ker f \oplus U$. The restriction $f|_U : U \to \operatorname{im} f$ is an isomorphism. Choose also $V_2 = \operatorname{im} f \oplus C$. Then f is the direct sum of the zero map $\ker f \to 0$, the isomorphism $U \to \operatorname{im} f$, and the zero map $0 \to C$. Choosing bases gives the displayed decomposition. Uniqueness follows because $\dim\ker f$, $\operatorname{rank} f$, and $\dim\operatorname{coker} f$ are isomorphism invariants. □

2. The existential quotient of the multiplicity cone

Following Proposition 1.2, let the multiplicity cone be $C = \mathbb{N}^3$ and define

$$\pi \colon C \longrightarrow \mathbb{N}^2, \qquad \pi(u, k, v) = (u + k, k + v).$$

This is the map that sends a representation to its dimension vector.

Definition 2.1 (Ray-deletion graph)**.** There is an edge from x to y in C if $x - y$ is one of $r(1, 0, 0)$, $r(0, 1, 0)$, or $r(0, 0, 1)$, where $r \geq 1$. In the *existential quotient* along π, there is an edge from d to d' if there exist $x, y \in C$ such that $\pi(x) = d$, $\pi(y) = d'$, and $x \to y$.

Theorem 2.2 (Recovery of the Wythoff graph)**.** *The existential quotient of the ray-deletion graph along π is the Wythoff graph.*

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Proof. Deleting r copies from the first, second, or third coordinate of C decreases the dimension vector by $r(1,0)$, $r(1,1)$, or $r(0,1)$, respectively. Thus every quotient edge is a Wythoff move.

Conversely, a move decreasing only the first coordinate of (a,b) by r is lifted by $(a,0,b) \to (a-r,0,b)$; the second-coordinate case is analogous. A move decreasing both coordinates by r is lifted by

$$(a-r, r, b-r) \longrightarrow (a-r, 0, b-r).$$

The displayed triples are nonnegative whenever the corresponding Wythoff move is legal. □

3. The Boolean Hall shadow

For the relation between Hall algebras and quiver representations, see Ringel [4]; for an introduction from the definitions, see Schiffmann [5]. Because the ground field is finite, the following numbers are finite. For representations U, V, X, define the Hall number

$$g^X_{U,V} = \#\{, W \subseteq X \mid W \cong V, X/W \cong U, \}.$$

Let $\mathbb{B} = \{0,1\}$ be the Boolean semiring. On the free $\mathbb{B}$-semimodule with basis the isomorphism classes of representations, define

$$[U] \star [V] = \bigvee_{g^X_{U,V} > 0} [X].$$

For finite sums, this is the ordinary Hall product with every coefficient replaced by its zero/nonzero value. For a support A, define its dimension-vector shadow by

$$D(A) = \{\mathbf{dim} X \mid [X] \text{ occurs in } A\}.$$

Lemma 3.1. *For any representations U, V,*

$$D([U] \star [V]) = \{\mathbf{dim} U + \mathbf{dim} V\}.$$

Proof. If $g^X_{U,V} > 0$, there is a short exact sequence $0 \to V \to X \to U \to 0$, hence $\mathbf{dim} X = \mathbf{dim} U + \mathbf{dim} V$. Conversely, the split sequence $0 \to V \to U \oplus V \to U \to 0$ exists, so the support of the product is nonempty. □

Theorem 3.2 (Hall shadow of Wythoff moves)**.** *For $d, d' \in \mathbb{N}^2$, there is a Wythoff move from d to d' if and only if there exist $U \in \mathrm{Rep}_{\mathbb{F}_q}(Q)$, $r \geq 1$, and $T \in \{S_1, S_2, E\}$ such that*

$$\mathbf{dim} U = d', \qquad d \in D([U] \star [T^{\oplus r}]).$$

Proof. By Lemma 3.1, the right-hand side is equivalent to $d - d' = r\,\mathbf{dim} T$. The dimension vectors of the three choices of T are $(1,0)$, $(0,1)$, and $(1,1)$. This is exactly Definition 1.1. A representation U of any prescribed dimension vector d' exists, for example as a zero map between vector spaces of the required dimensions. □

Remark 3.3 (Limitation). In Theorem 3.2, the dimension-vector shadow forgets extension classes, the values of the Hall numbers, and the field size q. Consequently, this correspondence alone gives no formula for Grundy values, no control of the mex recursion, and no faster algorithm. What has been proved here is exactly a representation of the legal-move relation.

References

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