A Literature Note on the Finitely Supported Commutative-Monoid-Valued Functor F_A

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Abstract

For a commutative monoid A, consider the finitary endofunctor

$$F_A(X) = \{f : X \to A \mid \mathrm{supp}(f) \text{ is finite}\},$$
$$\mathrm{supp}(f) = \{x \in X \mid f(x) \neq 1_A\}.$$

The purpose of this note is to place this functor in the literature from three perspectives:

(i) the perspective of viewing it as an object of $[\mathrm{FinSet}, \mathrm{Set}]$;

(ii) the perspective of n-skeletal and n-coskeletal objects; and

(iii) an explicit description of natural transformations $F_A \Rightarrow F_B$.

Among the references examined, the most directly relevant recent work is Kori–Watanabe [1]. They define precisely the branching functor arising from a commutative monoid and give an explicit description of $\mathrm{Nat}(F_A, F_B)$ when the source monoid is singly generated. Dahlqvist–Neves [3] provides an earlier treatment of special cases of multiset and finite-powerset type. This note is not intended as an exhaustive bibliography; it prioritizes the works closest to the question at hand.

1 Basic setup and the perspective from $[\mathrm{FinSet}, \mathrm{Set}]$

For a commutative monoid A, define

$$F_A(X) = \{f : X \to A \mid \mathrm{supp}(f) \text{ is finite}\}.$$

For a finite set S, the finite-support condition is automatic, and hence

$$F_A(S) = A^S.$$

For a map $u : S \to T$, the action is given by

$$(F_A(u)(f))(t) = \prod_{u(s)=t} f(s).$$

The expression on the right is well defined because A is commutative.

This functor is finitary. If

$$i : \mathrm{FinSet} \hookrightarrow \mathrm{Set}$$

denotes the inclusion, then it can be reconstructed as the left Kan extension

$$F_A \cong \mathrm{Lan}_i(F_A|_{\mathrm{FinSet}}).$$

Thus, the essential data of F_A may be regarded as the object

$$\mathrm{FinSet} \to \mathrm{Set}, \qquad S \mapsto A^S$$

of $[\mathrm{FinSet}, \mathrm{Set}]$. The category $[\mathrm{FinSet}, \mathrm{Set}]$ occurs as the classifying topos of the theory of objects [8, 9, 16]. In this sense, F_A may be regarded as an object of the object classifier.

From this perspective, for every set X,

$$F_A(X) \cong \varinjlim_{S \subseteq_{\mathrm{fin}} X} A^S.$$

This can be identified with the finitary endofunctor $X \mapsto X \otimes F_A$ corresponding to the object F_A of $[\mathrm{FinSet}, \mathrm{Set}]$ [8, 9, 16].

2 Claims to be compared with the literature

The claims to be situated in the literature can be summarized as follows.

Proposition 1 (Claims under comparison)**.** *Let A and B be commutative monoids.*

(1) If $A \neq 1$, then F_A is not n-skeletal for any finite n.

(2) If $A \neq 1$, then F_A is 3-coskeletal but not 2-coskeletal.

(3) Natural transformations $\eta : F_A \Rightarrow F_B$ are in one-to-one correspondence with maps

$$p : A \times A \to B$$

satisfying

$$p(1_A, c) = 1_B, \qquad p(ab, c) = p(a, bc)p(b, ac).$$

The correspondence is given by

$$(\eta_X(f))(x) = p\left(f(x), \prod_{y \neq x} f(y)\right).$$

Remark 1. *Claims (2) and (3) fit together closely. Indeed, if* $j_3 : \mathrm{FinSet}_{\leq 3} \hookrightarrow \mathrm{FinSet}$*, then the full subcategory of* 3*-coskeletal objects is reflective, with reflector*

$$\mathrm{cosk}_3 = \mathrm{Ran}_{j_3} j_3^*.$$

Consequently, if the target is 3*-coskeletal, a natural transformation is completely determined by its data on sets of cardinalities* $0, 1, 2, 3$*. The first component on a two-element set,*

$$p(a, b) = (\eta_{[2]}(a, b))(0),$$

is the basic datum, while compatibility on a three-element set forces exactly the relation

$$p(ab, c) = p(a, bc)p(b, ac).$$

This provides the conceptual explanation.

Remark 2. *At least among the references checked for this note, claim (3) in the form above does not appear to have been stated and proved for arbitrary commutative monoids* A *and* B*. The closest result is [1], which describes* $\mathrm{Nat}(F_A, F_B)$ *rather concretely when the source monoid is singly generated.*

3 The closest references

3.1 Kori–Watanabe

The recent reference closest to the present topic is Kori–Watanabe [1]. The following points are particularly relevant.

(a) They explicitly define the functor F_A arising from a commutative monoid A.

(b) Their examples include the multiset functor for $A = \mathbb{N}$ and the covariant finite-powerset functor for $A = \{0, 1\}$ equipped with $\vee$.

(c) One of their main topics is a characterization of the natural transformations $\mathrm{Nat}(F_A, F_B)$.

(d) In particular, Theorem 4.1 explicitly describes $\mathrm{Nat}(F_A, F_B)$ when the source monoid A is singly generated.

(e) In their conclusion, extending the characterization to finitely generated source monoids is identified as a direction for future work.

Therefore, if the general formula

$$\mathrm{Nat}(F_A, F_B) \cong \{p : A \times A \to B \mid p(1_A, c) = 1_B,\ p(ab, c) = p(a, bc)p(b, ac)\}$$

is correct, it can be read as extending the singly generated case of [1] to arbitrary commutative monoids.

The surrounding framework used in that paper is itself supported by the general theory of coalgebraic product constructions developed by Watanabe–Junges–Rot–Hasuo [2].

3.2 Dahlqvist–Neves

Dahlqvist–Neves [3] classifies natural transformations

$$1 \to T, \qquad T \times T \to T$$

for several monads T. Their examples include the multiset monad, and Kori–Watanabe explicitly traces the known characterizations for the finite-powerset and multiset cases back to Dahlqvist–Neves [1].

The main focus of [3], however, is the classification of monadic binary operations. It does not directly provide a complete classification of

$$\mathrm{Nat}(F_A, F_B)$$

for arbitrary commutative monoids A and B. Thus, for the present question, [3] is better regarded as an important precursor for special cases than as the direct prior reference.

4 Background literature: coalgebra and set functors

4.1 The context of monoid-valued functors

Gumm–Schröder [4] studies monoid-labeled transition systems and shows that finitely supported value functors arising from commutative monoids—or, more generally, from suitable algebraic data—are natural from a coalgebraic point of view. It is a primary reference for presenting the present F_A as a basic example of weighted or monoid-valued branching.

4.2 The context of accessible and finitary set functors

Adámek–Gumm–Trnková [5] is a classical reference that gives presentations of accessible set functors. Since F_A is finitary, this line of work is useful background for explaining why restriction to FinSet is sufficient. Adámek–Trnková [6] is also useful for the classical background.

5 Background literature: classifying topoi, Lawvere theories, and finitary functors

The perspective in which $[\mathrm{FinSet}, \mathrm{Set}]$ is the classifying topos of the theory of objects goes back to Mac Lane–Moerdijk [8]; Borceux [9] also explains it in terms of finite-limit theories and Kan extensions. Foundational references for the general theory of finitary monads and Lawvere theories include Kelly–Power [10], Lack–Rosický [11], and Adámek–Rosický [7].

In the present context, this general theory has two uses:

(1) because F_A is finitary, it can be reconstructed from its values A^S and its action on morphisms in FinSet;

(2) for F_A viewed as an object of $[\mathrm{FinSet}, \mathrm{Set}]$, evaluation at a set X can be written as the coend or tensor

$$X \otimes F_A.$$

From this perspective, skeleta and coskeleta are understood as Kan extensions along

$$j_n : \mathrm{FinSet}_{\leq n} \hookrightarrow \mathrm{FinSet}.$$

This makes the discussion of 3-coskeletality quite natural.

6 Background literature: levels, skeleta, coskeleta, and Aufhebung

The general theory of skeleta, coskeleta, and levels originates in Lawvere's questions and has been substantially organized in a series of recent works by Menni [14, 15]. A classical reference with important computations is Kennett–Riehl–Roy–Zaks [13], which establishes sharp relationships between n-skeletal and m-coskeletal objects in simplicial and cubical sets.

Hora–Kamio–Maehara [17] provides a recent computation for symmetric simplicial sets. Its site is not the present $[\mathrm{FinSet}, \mathrm{Set}]$, but it is close in spirit as a computation of levels in a presheaf or functor category based on finite sets. It is therefore natural to read the claim

$$F_A \text{ is 3-coskeletal}$$

within this line of work.

7 Provisional positioning

On the basis of this literature search, the provisional positioning is roughly as follows.

(1) The functor

$$F_A(X) = \{f : X \to A \mid \operatorname{supp}(f) \text{ is finite}\}$$

is itself classical and is known in the context of coalgebra and weighted transition systems [4, 1].

(2) The perspective of viewing it as an object of [FinSet, Set] is also classical and fits the general theory of the classifying topos of objects, finitary functors, and Kan extensions [8, 9, 10, 11, 7, 16].

(3) By contrast, the explicit statement

$$A \neq 1 \implies F_A \text{ is 3-coskeletal but not 2-coskeletal}$$

does not appear, at least in the literature checked for this note, in exactly this form.

(4) Likewise, the statement that

$$\mathrm{Nat}(F_A, F_B)$$

for arbitrary commutative monoids A and B can be described completely by the binary datum

$$p : A \times A \to B$$

and the cocycle-like relation

$$p(ab, c) = p(a, bc)p(b, ac)$$

was not found explicitly among the references checked.

(5) The closest published or preprint context is [1], which treats the case of a singly generated source monoid and leaves the finitely generated case as future work. Thus, the present general form is naturally positioned as a candidate result directly extending [1].

8 How to divide the citations in practice

In a paper or note, the references can conveniently be divided as follows.

(1) the definition of F_A itself and its coalgebraic context:

[4, 1];

(2) natural transformations in special cases of multiset and finite powerset type:

[3, 1];

(3) [FinSet, Set], the classifying topos of objects, and Kan extensions:

[8, 9, 16];

(4) the general theory and computational examples of levels, skeleta, coskeleta, and Aufhebung:

[12, 13, 14, 15, 17].

In summary, the present topic combines three strands:

$$\begin{aligned} &\text{coalgebraic branching functors} \\ +\ &\text{the classifying-topos perspective on } [\mathrm{FinSet}, \mathrm{Set}] \\ +\ &\text{levels and coskeleta.} \end{aligned}$$

The closest prior work is Kori–Watanabe [1]. Their untreated finitely generated or general commutative-monoid case fits precisely with the present description based on 3-coskeletality. This is the most natural understanding at the present stage.

References

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