THE LOCAL STATE CLASSIFIER OF A SLICE TOPOS

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Abstract. Let $\mathcal{E}$ be a topos possessing a local state classifier $\Xi = \operatorname{colim}(\mathcal{E}_{\mathrm{mono}} \hookrightarrow \mathcal{E})$, with universal cocone $\{\xi_X : X \to \Xi\}$, and let X be an object of $\mathcal{E}$. We prove that the slice topos $\mathcal{E}/X$ again possesses a local state classifier, and that it is the *principal down-set object*

$$D_X \;=\; \{\, (s,x) \in \Xi \times X \;:\; s \leq \xi_X(x) \,\},$$

that is, the pullback along ξ_X of the projection $\preceq \to \Xi$ onto the *larger* coordinate, regarded as an object of $\mathcal{E}/X$ via the second projection. Informally: the local states of $\mathcal{E}/X$ are exactly the local states of $\mathcal{E}$ that are at least as unfolded as X itself. The proof is elementary and site-free; it uses only the folding inequality $\xi_A \leq \xi_Y \circ f$, the product formula $\xi_{A\times B} = \xi_A \wedge \xi_B$, and the retraction $(s,x) \mapsto (s \wedge \xi_X(x), x)$ of $\Xi \times X$ onto D_X. In particular the existence of $\Xi_{\mathcal{E}/X}$ is not assumed but proved. We record the induced semilattice structure, verify the formula on group actions and on localic toposes, observe that the naive analogue of the slice-stability $\Omega_{\mathcal{E}/X} \cong X^*\Omega$ of the subobject classifier *fails* for Ξ, and give a second, site-theoretic proof for presheaf toposes.

1. Introduction

The subobject classifier is stable under slicing: for a topos $\mathcal{E}$ and an object X, the projection $\Omega \times X \to X$ is the subobject classifier of $\mathcal{E}/X$, i.e. $\Omega_{\mathcal{E}/X} \cong X^*\Omega$ [4, IV.7]. The *local state classifier* Ξ, introduced in [2] to give an internal parameterization of hyperconnected quotients analogous to the parameterization of subtoposes by Lawvere–Tierney topologies, is a colimit rather than a limit, and one should not expect it to be stable under slicing in the same naive way. Indeed it is not (Example 6.3).

The purpose of this note is to prove the correct formula. Write $\preceq \rightarrowtail \Xi \times \Xi$ for the order relation $\{(s,t) : s \leq t\}$ of the internal meet-semilattice Ξ, with the first coordinate the smaller one, and let $\xi_X : X \to \Xi$ be the canonical morphism.

Theorem 1.1 (Theorem 4.1)**.** *Let $\mathcal{E}$ be a topos with a local state classifier Ξ, and let $X \in \mathrm{ob}(\mathcal{E})$. Then $\mathcal{E}/X$ has a local state classifier, and it is*

$$\Xi_{\mathcal{E}/X} \;\cong\; D_X \;:=\; X \times_\Xi \preceq$$

— the pullback of the projection $\mathrm{pr}_2 :\ \preceq \to \Xi$ onto the larger coordinate along $\xi_X : X \to \Xi$, regarded as an object of $\mathcal{E}/X$ by the projection $\pi : D_X \to X$. The universal cocone is $\lambda_p = \langle \xi_A, p\rangle : A \to D_X$ for $p : A \to X$ in $\mathcal{E}/X$.

The fibre of D_X over a (generalized) point x of X is the principal down-set $\downarrow\xi_X(x)$: the local states more unfolded than the local state of x. The orientation of the projection is essential; taking the other coordinate produces a wrong answer already for $\mathcal{E} = \mathrm{Set}^{G^{\mathrm{op}}}$ (Remark 4.2).

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Two features of the proof deserve emphasis. First, it is *site-free*: it takes place in the internal language of $\mathcal{E}$ and uses only the two structural facts about Ξ recalled in Section 2 together with the retraction $r(s,x) = (s \wedge \xi_X(x), x)$. Second, it does *not* presuppose the existence of $\Xi_{\mathcal{E}/X}$; the object D_X is exhibited directly as a colimit, so the theorem is simultaneously an existence statement. This matters because sheafification does not in general preserve local state classifiers [2, Rem. 3.24], so existence statements for Ξ are not cheap.

Section 2 fixes notation and recalls what we use from [2]. Section 3 constructs D_X and its cocone. Section 4 proves the theorem. Section 5 records the induced semilattice structure and a localicity criterion. Section 6 computes examples. Section 7 gives an independent site-theoretic proof for presheaf toposes. Section 8 discusses related work, and Section 9 lists what remains open.

Provenance and status. This memorandum is written from public sources only. The mathematical content of Sections 2–6 has been checked by two independent reviews; the statements of Section 7 concerning sheaf toposes are explicitly labelled as a sketch. The author's unpublished notes *Notes on advances of LSC* have not been collated with this draft, and may well contain part of what is proved here.

2. Preliminaries

Throughout, $\mathcal{E}$ is an elementary topos. We freely use generalized elements: an expression such as "(s,x) with $s \leq \xi_X(x)$" abbreviates the corresponding statement about morphisms $Z \to \Xi \times X$, and all such computations are valid in the internal language.

2.1. **Local state classifiers.** Let $\mathcal{C}$ be a category and $\mathcal{C}_{\mathrm{mono}} \subseteq \mathcal{C}$ the wide subcategory whose morphisms are the monomorphisms of $\mathcal{C}$.

Definition 2.1 ([2, §3])**.** A *locally determined cocone* on $\mathcal{C}$ is a cocone under the inclusion functor $\mathcal{C}_{\mathrm{mono}} \hookrightarrow \mathcal{C}$: a family $\{\psi_Z \colon Z \to T\}_{Z \in \mathrm{ob}(\mathcal{C})}$ such that $\psi_Y \circ \iota = \psi_Z$ for every monomorphism $\iota \colon Z \rightarrowtail Y$. A *local state classifier* of $\mathcal{C}$ is a colimit Ξ of $\mathcal{C}_{\mathrm{mono}} \hookrightarrow \mathcal{C}$, if it exists; its universal cocone is written $\{\xi_Z \colon Z \to \Xi\}_{Z \in \mathrm{ob}(\mathcal{C})}$.

Note that no compatibility with non-monic morphisms is required: this is exactly what makes Ξ a nontrivial invariant. Every Grothendieck topos has a local state classifier [2, §3.4], constructed as the sheafification $\mathbf{a}\Xi_0$ of an explicit presheaf Ξ_0 (see Section 7). For elementary toposes the existence is not known in general and we take it as a hypothesis.

We use two structural facts about Ξ.

Fact 2.2 ([2, Prop. 3.27])**.** If $\mathcal{E}$ is a topos with local state classifier Ξ, then Ξ carries a canonical structure of an internal meet-semilattice, with top element $\top := \xi_1 \colon 1 \to \Xi$ and meet $\wedge \colon \Xi \times \Xi \to \Xi$ uniquely determined by the *product formula*

$$\xi_{A \times B} \;=\; \wedge \circ (\xi_A \times \xi_B) \colon A \times B \longrightarrow \Xi, \qquad A, B \in \mathrm{ob}(\mathcal{E}). \tag{1}$$

(In [2] the meets $\wedge_n \colon \Xi^n \to \Xi$ are defined for all n by the analogous n-ary formula; $\top = \wedge_0$ and $\wedge = \wedge_2$.)

Definition 2.3. For $a, b \colon Z \to \Xi$ put $a \leq b$ iff $a = \wedge \circ \langle a, b \rangle$. By Fact 2.2 this is a partial order on $\mathcal{E}(Z, \Xi)$, natural in Z. We write

$$\preceq \;:=\; \mathrm{Eq}\big(\mathrm{pr}_1,\ \wedge \colon \Xi \times \Xi \rightrightarrows \Xi\big) \;\rightarrowtail\; \Xi \times \Xi,$$

so that a morphism $\langle a, b \rangle \colon Z \to \Xi \times \Xi$ factors through $\preceq$ if and only if $a \leq b$. Thus $\preceq\, = \{(s,t) : s \leq t\}$: the *first* coordinate is the smaller one.

2.2. Monomorphisms in a slice.

Lemma 2.4. *Let $\mathcal{C}$ be a category, $X \in \mathrm{ob}(\mathcal{C})$, and $\iota\colon (A,p) \to (B,q)$ a morphism of $\mathcal{C}/X$. Then ι is a monomorphism in $\mathcal{C}/X$ if and only if ι is a monomorphism in $\mathcal{C}$.*

Proof. If ι is monic in $\mathcal{C}$ it is monic in $\mathcal{C}/X$, since $\mathcal{C}/X(Z,-)$ is a subset of $\mathcal{C}(Z,-)$. Conversely let ι be monic in $\mathcal{C}/X$ and let $u, v\colon Z \to A$ in $\mathcal{C}$ satisfy $\iota u = \iota v$. Then $pu = q\iota u = q\iota v = pv =: w$, so $u, v\colon (Z,w) \to (A,p)$ are morphisms of $\mathcal{C}/X$ with $\iota u = \iota v$, whence $u = v$. □

Consequently a locally determined cocone on $\mathcal{E}/X$ is a family $\{\psi_p\colon A \to T\}$ indexed by the objects $p\colon A \to X$ of $\mathcal{E}/X$, with $\tau \circ \psi_p = p$ for the structure map $\tau\colon T \to X$, such that $\psi_q \circ \iota = \psi_p$ for every ι that is monic in $\mathcal{E}$ and satisfies $q\iota = p$.

2.3. The folding inequality.

Lemma 2.5 ([2, Lem. 4.9])**.** *For every morphism $f\colon A \to Y$ of $\mathcal{E}$ one has $\xi_A \le \xi_Y \circ f$.*

Proof. The graph $\Gamma_f = \langle \mathrm{id}_A, f\rangle\colon A \to A \times Y$ is split by pr_1, hence monic, so $\xi_{A\times Y} \circ \Gamma_f = \xi_A$ by Definition 2.1. Combining with (1),

$$\xi_A = \xi_{A\times Y} \circ \Gamma_f = \wedge \circ (\xi_A \times \xi_Y) \circ \langle \mathrm{id}_A, f\rangle = \wedge \circ \langle \xi_A,\ \xi_Y \circ f\rangle,$$

which is precisely $\xi_A \le \xi_Y \circ f$. □

Menni [5] calls the resulting structure a *lax cocone*, and uses the case of equality to define non-singular maps.

2.4. Joint epimorphy.

Lemma 2.6. *Let Ξ be a local state classifier of $\mathcal{E}$.*

1. *If $g, h\colon \Xi \to Y$ satisfy $g \circ \xi_A = h \circ \xi_A$ for all $A \in \mathrm{ob}(\mathcal{E})$, then $g = h$.*
2. *If $g, h\colon \Xi \times X \to Y$ satisfy $g \circ (\xi_A \times \mathrm{id}_X) = h \circ (\xi_A \times \mathrm{id}_X)$ for all $A \in \mathrm{ob}(\mathcal{E})$, then $g = h$.*

Proof. (1) The family $\{g \circ \xi_A\}_A$ is a locally determined cocone with vertex Y, and both g and h factor the universal cocone through it; uniqueness in Definition 2.1 gives $g = h$. (2) Transposing along the adjunction $(-) \times X \dashv (-)^X$ turns the hypothesis into $\hat{g} \circ \xi_A = \hat{h} \circ \xi_A$ for $\hat{g}, \hat{h}\colon \Xi \to Y^X$, and (1) applies. □

3. The principal down-set object

Fix a topos $\mathcal{E}$ with local state classifier Ξ and an object $X \in \mathrm{ob}(\mathcal{E})$.

Definition 3.1. Let D_X be the pullback

$$\begin{tikzcd} D_X \arrow[r] \arrow[d, "\pi"'] & \preceq \arrow[d, "\mathrm{pr}_2"] \\ X \arrow[r, "\xi_X"'] & \Xi \end{tikzcd}$$

of the projection onto the *larger* coordinate along ξ_X. We regard D_X as an object of $\mathcal{E}/X$ via π.

Lemma 3.2. *D_X is canonically a subobject $\iota\colon D_X \rightarrowtail \Xi \times X$, namely the equalizer of*

$$\mathrm{pr}_1, \quad \wedge \circ (\mathrm{id}_\Xi \times \xi_X)\ :\ \Xi \times X \rightrightarrows \Xi,$$

and under this identification $\pi = \mathrm{pr}_2 \circ \iota$. Thus, on generalized elements,

$$D_X = \{(s,x) \in \Xi \times X\ :\ s \le \xi_X(x)\},$$

whose fibre over x is the principal down-set ${\downarrow}\xi_X(x)$.

Proof. Both objects represent the same subfunctor of $\mathcal{E}(-, \Xi \times X)$: a morphism $\langle a, x\rangle \colon Z \to \Xi \times X$ factors through the stated equalizer iff $a \leq \xi_X \circ x$, and it factors through the pullback of Definition 3.1 iff $\langle a, \xi_X \circ x\rangle$ factors through $\preceq$, i.e. iff $a \leq \xi_X \circ x$. $\square$

Definition 3.3. For an object $p\colon A \to X$ of $\mathcal{E}/X$ put

$$\lambda_p \;:=\; \langle \xi_A,\ p\rangle \colon A \longrightarrow \Xi \times X.$$

By Lemma 2.5, $\xi_A \leq \xi_X \circ p$, so λ_p factors uniquely through ι; we denote the factorization again by $\lambda_p \colon A \to D_X$. Since $\pi \circ \lambda_p = p$, it is a morphism $(A, p) \to (D_X, \pi)$ of $\mathcal{E}/X$.

Lemma 3.4. *$\{\lambda_p\}_{(A,p)\in \mathrm{ob}(\mathcal{E}/X)}$ is a locally determined cocone on $\mathcal{E}/X$ with vertex (D_X, π).*

Proof. Let $\iota\colon (A, p) \to (B, q)$ be monic in $\mathcal{E}/X$. By Lemma 2.4 it is monic in $\mathcal{E}$, so $\xi_B \circ \iota = \xi_A$; and $q \circ \iota = p$. Hence $\lambda_q \circ \iota = \langle \xi_B \iota,\ q\iota\rangle = \langle \xi_A, p\rangle = \lambda_p$. $\square$

The following retraction is the engine of the proof.

Lemma 3.5. *Let $r := \langle \wedge \circ (\mathrm{id}_\Xi \times \xi_X),\ \mathrm{pr}_2\rangle \colon \Xi \times X \to \Xi \times X$, i.e. $r(s, x) = (s \wedge \xi_X(x),\ x)$. Then:*

1. *r factors through ι, giving $r\colon \Xi \times X \to D_X$ with $\pi \circ r = \mathrm{pr}_2$; in particular r is a morphism $X^*\Xi = (\Xi \times X, \mathrm{pr}_2) \to (D_X, \pi)$ of $\mathcal{E}/X$;*
2. *$r \circ \iota = \mathrm{id}_{D_X}$, so ι is a split monomorphism and r a split epimorphism in $\mathcal{E}/X$;*
3. *for every $A \in \mathrm{ob}(\mathcal{E})$,*
$$r \circ (\xi_A \times \mathrm{id}_X) \;=\; \lambda_{\mathrm{pr}_2} \colon A \times X \longrightarrow D_X, \tag{2}$$
where on the right $\mathrm{pr}_2\colon A \times X \to X$ is viewed as an object of $\mathcal{E}/X$.

Proof. (1) By idempotence and associativity of $\wedge$, $(s \wedge \xi_X(x)) \wedge \xi_X(x) = s \wedge \xi_X(x)$, so the first component of r is $\leq \xi_X \circ \mathrm{pr}_2$; apply Lemma 3.2. The second component being pr_2 gives $\pi \circ r = \mathrm{pr}_2$.

(2) On D_X we have $s \wedge \xi_X(x) = s$ by definition, so $r(s, x) = (s, x)$.

(3) Both sides have second component pr_2. The first component of the left-hand side is $\wedge \circ (\xi_A \times \xi_X) = \xi_{A\times X}$ by the product formula (1), which is the first component of $\lambda_{\mathrm{pr}_2} = \langle \xi_{A\times X}, \mathrm{pr}_2\rangle$. $\square$

4. The main theorem

Theorem 4.1. *Let $\mathcal{E}$ be a topos possessing a local state classifier Ξ, and let $X \in \mathrm{ob}(\mathcal{E})$. Then the slice topos $\mathcal{E}/X$ possesses a local state classifier, and*

$$\big(\Xi_{\mathcal{E}/X},\ \{\xi^{\mathcal{E}/X}_{(A,p)}\}\big) \;=\; \big((D_X, \pi),\ \{\lambda_p\}\big).$$

Explicitly, $\Xi_{\mathcal{E}/X} \cong X \times_\Xi \preceq$, the pullback along ξ_X of the projection $\preceq \to \Xi$ onto the larger coordinate.

Proof. By Lemma 3.4 it remains to prove the universal property. Let $\{\psi_p \colon A \to T\}_{(A,p)}$ be a locally determined cocone on $\mathcal{E}/X$ with vertex (T, τ).

Existence. For $A \in \mathrm{ob}(\mathcal{E})$ consider the object $(A \times X, \mathrm{pr}_2)$ of $\mathcal{E}/X$ and put

$$\varphi_A \;:=\; \psi_{\mathrm{pr}_2} \colon A \times X \longrightarrow T, \qquad \widehat{\varphi_A}\colon A \longrightarrow T^X$$

for its exponential transpose. We claim $\{\widehat{\varphi_A}\}_{A\in\mathrm{ob}(\mathcal{E})}$ is a locally determined cocone on $\mathcal{E}$ with vertex T^X. Indeed, if $\kappa\colon U \rightarrowtail A$ is monic in $\mathcal{E}$, then $\kappa \times \mathrm{id}_X \colon U \times X \to A \times X$ is monic in $\mathcal{E}$ and commutes with the second projections, hence is monic in $\mathcal{E}/X$ by Lemma 2.4; the cocone property of ψ gives $\psi_{\mathrm{pr}_2} \circ (\kappa \times \mathrm{id}_X) = \psi_{\mathrm{pr}_2}$, i.e. $\varphi_A \circ (\kappa \times \mathrm{id}_X) = \varphi_U$, and transposing, $\widehat{\varphi_A} \circ \kappa = \widehat{\varphi_U}$.

By the universal property of Ξ there is therefore a unique $v\colon \Xi \to T^X$ with $v \circ \xi_A = \widehat{\varphi_A}$ for all A. Let $\bar{v}\colon \Xi \times X \to T$ be its transpose, so that

$$\bar{v} \circ (\xi_A \times \mathrm{id}_X) \;=\; \psi_{\mathrm{pr}_2}\colon A \times X \to T, \qquad A \in \mathrm{ob}(\mathcal{E}). \tag{3}$$

First, $\bar{v}$ is a morphism over X. Indeed $\tau \circ \bar{v}$ and pr_2 are two morphisms $\Xi \times X \to X$, and by (3), $\tau \circ \bar{v} \circ (\xi_A \times \mathrm{id}_X) = \tau \circ \psi_{\mathrm{pr}_2} = \mathrm{pr}_2 = \mathrm{pr}_2 \circ (\xi_A \times \mathrm{id}_X)$ for every A; Lemma 2.6(2) gives $\tau \circ \bar{v} = \mathrm{pr}_2$.

Now set $u := \bar{v} \circ \iota\colon D_X \to T$; by the previous paragraph $\tau \circ u = \mathrm{pr}_2 \circ \iota = \pi$, so u is a morphism $(D_X, \pi) \to (T, \tau)$ of $\mathcal{E}/X$. Let $p\colon A \to X$ be an object of $\mathcal{E}/X$. The graph $\Gamma_p = \langle \mathrm{id}_A, p\rangle\colon A \to A \times X$ satisfies $\mathrm{pr}_2 \circ \Gamma_p = p$ and is split by pr_1, hence is a monomorphism $(A, p) \rightarrowtail (A \times X, \mathrm{pr}_2)$ of $\mathcal{E}/X$ by Lemma 2.4. Therefore $\psi_{\mathrm{pr}_2} \circ \Gamma_p = \psi_p$. Since $\iota \circ \lambda_p = \langle \xi_A, p\rangle = (\xi_A \times \mathrm{id}_X) \circ \Gamma_p$, we conclude

$$u \circ \lambda_p = \bar{v} \circ (\xi_A \times \mathrm{id}_X) \circ \Gamma_p \overset{(3)}{=} \psi_{\mathrm{pr}_2} \circ \Gamma_p = \psi_p.$$

Uniqueness. Let $u, u'\colon (D_X, \pi) \to (T, \tau)$ satisfy $u \circ \lambda_p = u' \circ \lambda_p$ for every object p of $\mathcal{E}/X$. Applying this to $p = \mathrm{pr}_2\colon A \times X \to X$ and using (2),

$$u \circ r \circ (\xi_A \times \mathrm{id}_X) = u \circ \lambda_{\mathrm{pr}_2} = u' \circ \lambda_{\mathrm{pr}_2} = u' \circ r \circ (\xi_A \times \mathrm{id}_X)$$

for every $A \in \mathrm{ob}(\mathcal{E})$. By Lemma 2.6(2), $u \circ r = u' \circ r$, and composing with ι and using Lemma 3.5(2) gives $u = u'$. □

Remark 4.2. The orientation of the projection $\preceq \to \Xi$ is essential. Pulling back along the *smaller* coordinate would produce $\{(t, x) : \xi_X(x) \leq t\}$, the principal *up*-set object; for $\mathcal{E} = \mathrm{Set}^{G^{\mathrm{op}}}$ and $X = G$ the regular action this has fibre $\{K \in \mathrm{Sub}(G) : \{e\} \subseteq K\} = \mathrm{Sub}(G)$, whereas $\mathcal{E}/G \simeq \mathrm{Set}$ has terminal local state classifier. See Example 6.1.

Remark 4.3. The proof used only: finite limits, cartesian closedness, the existence of Ξ with the structure of Fact 2.2, and Lemma 2.4. No site, no subobject classifier and no sheafification appear, and the existence of $\Xi_{\mathcal{E}/X}$ is a conclusion rather than a hypothesis. This is what makes the argument applicable to elementary toposes with a local state classifier, and not only to Grothendieck toposes.

5. Consequences

Corollary 5.1 (Semilattice structure)**.** *Under the identification of Theorem 4.1, the internal meet-semilattice structure of $\Xi_{\mathcal{E}/X}$ given by Fact 2.2 applied to $\mathcal{E}/X$ is the fibrewise one inherited from Ξ:*

$$(s, x) \wedge (s', x) = (s \wedge s',\, x), \qquad \top_{\mathcal{E}/X} = \lambda_{\mathrm{id}_X} = \langle \xi_X, \mathrm{id}_X\rangle\colon X \to D_X.$$

Proof. The top element is $\xi^{\mathcal{E}/X}_{1_{\mathcal{E}/X}}$, and $1_{\mathcal{E}/X} = (X, \mathrm{id}_X)$, so it is $\lambda_{\mathrm{id}_X} = \langle \xi_X, \mathrm{id}_X\rangle$, i.e. $x \mapsto (\xi_X(x), x)$. For the meet, the product of (A, p) and (B, q) in $\mathcal{E}/X$ is $A \times_X B$ with structure map $p \circ \mathrm{pr}_1$. The canonical morphism $A \times_X B \to A \times B$ is the pullback of the diagonal $X \rightarrowtail X \times X$ along $p \times q$, hence monic, so $\xi_{A \times_X B} = \xi_{A \times B}|_{A \times_X B} = (\xi_A \wedge \xi_B)|_{A \times_X B}$ by (1). Therefore $\lambda_{p\,\mathrm{pr}_1}(a, b) = (\xi_A(a) \wedge \xi_B(b),\, p(a))$, which is the fibrewise meet of $\lambda_p(a) = (\xi_A(a), p(a))$ and $\lambda_q(b) = (\xi_B(b), q(b))$ over the common base point. By Fact 2.2 this determines $\wedge_{\mathcal{E}/X}$. □

Corollary 5.2. *Let $\mathcal{E}$ be a Grothendieck topos with local state classifier Ξ and $X \in \mathrm{ob}(\mathcal{E})$. Then $\mathcal{E}/X$ is localic if and only if $\pi\colon D_X \to X$ is an isomorphism, i.e. if and only if $\top_{\mathcal{E}/X} = \langle \xi_X, \mathrm{id}_X\rangle$ is an isomorphism onto D_X.*

Proof. $\mathcal{E}/X$ is a Grothendieck topos, and a Grothendieck topos is localic if and only if its local state classifier is terminal [2, Cor. 5.2]. The terminal object of $\mathcal{E}/X$ is (X, id_X), and by Theorem 4.1 the local state classifier is (D_X, π). □

6. Examples

Example 6.1 (Group actions)**.** Let G be a group and $\mathcal{E} = \mathrm{Set}^{G^{\mathrm{op}}}$ the topos of right G-sets. By [2, Ex. 3.10], $\Xi = \mathrm{Sub}(G)$, the set of all subgroups with the right conjugation action $H * g = g^{-1}Hg$, the semilattice order being inclusion, and $\xi_X(x) = \mathrm{Stab}_G(x)$.

Let $H \leq G$ and $X = H\backslash G$, the right G-set of right cosets. Then $\mathrm{Stab}_G(Hg) = \{k : Hgk = Hg\} = g^{-1}Hg$, so Theorem 4.1 gives

$$D_X = \{(K, Hg) \ : \ K \leq g^{-1}Hg\}, \qquad \pi(K, Hg) = Hg,$$

whose fibre over the base point H is $\{K : K \leq H\} = \mathrm{Sub}(H)$, carrying the H-conjugation action and ordered by inclusion.

On the other hand $\mathrm{Set}^{G^{\mathrm{op}}}/(H\backslash G) \simeq \mathrm{Set}^{H^{\mathrm{op}}}$, the equivalence sending $(A \to H\backslash G)$ to the fibre over the base point. An equivalence preserves and reflects monomorphisms, hence transports local state classifiers. And indeed $\Xi_{\mathrm{Set}^{H^{\mathrm{op}}}} = \mathrm{Sub}(H)$ with H-conjugation: the formula is confirmed.

The two extreme cases check out. For $H = G$ we get $X = 1$, $\mathcal{E}/X \simeq \mathcal{E}$ and $D_1 = \{(K, *) : K \leq G\} = \mathrm{Sub}(G) = \Xi$, as it must be since $\xi_1 = \top$. For $H = \{e\}$ we get $X = G$ with the regular action, $\mathcal{E}/X \simeq \mathrm{Set}$, and D_G has fibres $\mathrm{Sub}(\{e\}) = 1$, so $\pi \colon D_G \to G$ is an isomorphism — consistent with $\Xi_{\mathrm{Set}} = 1$ and with Corollary 5.2.

Example 6.2 (Localic toposes)**.** Let $\mathcal{E}$ be a localic Grothendieck topos. Then $\Xi = 1$ [2, Ex. 3.23, Cor. 5.2], so $\xi_X = ! \colon X \to 1$ and $\preceq = 1$ (as $\top \leq \top$), whence $D_X = X \times_1 1 = X$ and $\Xi_{\mathcal{E}/X} = 1_{\mathcal{E}/X}$. This agrees with the direct argument: $\mathcal{E}/X \to \mathcal{E}$ is étale, hence localic, and a composite of localic morphisms is localic, so $\mathcal{E}/X$ is localic and its local state classifier is terminal. The check is degenerate on both sides and therefore carries little information, but it does exclude sign errors of the kind discussed in Remark 4.2.

Example 6.3 (The naive slice-stability fails)**.** For the subobject classifier one has $\Omega_{\mathcal{E}/X} \cong X^*\Omega = (\Omega \times X, \mathrm{pr}_2)$ [4, IV.7]. The naive analogue $\Xi_{\mathcal{E}/X} \cong X^*\Xi$ is false. The smallest counterexample: take $G = \mathbb{Z}/2$ and $X = G$ with the regular action, so $\mathcal{E}/X \simeq \mathrm{Set}$ and $\Xi_{\mathcal{E}/X}$ has singleton fibres, whereas $X^*\Xi$ has fibres $\mathrm{Sub}(\mathbb{Z}/2)$, of cardinality 2. Theorem 4.1 explains the discrepancy: D_X is a proper subobject of $X^*\Xi$, cut out by the folding inequality, and equals $X^*\Xi$ only when ξ_X is constantly $\top$.

7. A second proof for presheaf toposes

For Grothendieck toposes there is an independent, site-theoretic route. We give it in full for presheaf toposes, where it is complete, and indicate what would be needed for sheaf toposes, where it is only a sketch.

Recall [2, §3.4] the presheaf Ξ_0 on a small category $\mathcal{C}$: $\Xi_0(c)$ is the set of co-subobjects (isomorphism classes of quotients) of the representable $y(c)$, and for a presheaf X the morphism $\xi_X \colon X \to \Xi_0$ sends $x \in X(c)$ to the class of the epimorphic part $y(c) \twoheadrightarrow \mathrm{Im}(\hat{x})$ of $\hat{x} \colon y(c) \to X$. For $\mathcal{E} = \mathrm{PSh}(\mathcal{C})$ this Ξ_0 is already the local state classifier; for $\mathcal{E} = \mathrm{Sh}(\mathcal{C}, J)$ one has $\Xi = \mathbf{a}\Xi_0$. The order is that of quotients, the terminal quotient $y(c) \twoheadrightarrow 1$ being the top element (consistently with $\top = \xi_1$).

Proposition 7.1. *Theorem 4.1 holds for $\mathcal{E} = \mathrm{PSh}(\mathcal{C})$, by direct computation on the site.*

Proof. Let $X \in \mathrm{PSh}(\mathcal{C})$ and $\mathrm{PSh}(\mathcal{C})/X \simeq \mathrm{PSh}(\int X)$, under which the representable at $(c,x) \in \int X$ corresponds to $\hat{x}\colon y(c) \to X$. Hence $\Xi_{\mathrm{PSh}(\int X)}(c,x)$ is the set of co-subobjects of $\hat{x}$ in $\mathrm{PSh}(\mathcal{C})/X$, i.e. of pairs $(q\colon y(c) \twoheadrightarrow Q,\ e\colon Q \to X)$ with $eq = \hat{x}$, up to isomorphism. Since q is epic, e is uniquely determined by q when it exists; so such pairs correspond bijectively to those co-subobjects q of $y(c)$ through which $\hat{x}$ factors. Now $\hat{x}$ factors through q iff $y(c) \twoheadrightarrow \mathrm{Im}(\hat{x})$ factors through q, i.e. iff $q \le \xi_X(x)$ in the order on $\Xi_0(c)$. Thus $\Xi_{\mathrm{PSh}(\int X)}(c,x) = {\downarrow}\xi_X(x)$.

On the other hand $\preceq$ is an equalizer, hence computed pointwise in a presheaf topos, so $(X \times_{\Xi_0} \preceq)(c) = \{(x,\zeta) : \zeta \le \xi_X(x)\}$, which is the same presheaf on $\int X$, with the same restriction maps and the same order. The identification is compatible with the cocones by construction. ☐

Remark 7.2 (The sheaf case)**.** For $\mathcal{E} = \mathrm{Sh}(\mathcal{C}, J)$ the same computation would descend along sheafification $\mathbf{a}$, comparing the two sides at the level of the presheaf approximations of [2, §3.4] and then using the left exactness of $\mathbf{a}$. Two routine but so far unverified checks remain:

(i) that the internal semilattice structure of $\Xi = \mathbf{a}\Xi_0$ of Fact 2.2 is the sheafification of the pointwise meet of co-subobjects on Ξ_0 (this is a straightforward calculation for presheaves [2, Ex. 3.28], but is not in the literature for sheaves), which would yield $\preceq = \mathbf{a}(\preceq_0)$ by left exactness;

(ii) that under $\mathrm{Sh}(\int X, J_X) \simeq \mathrm{Sh}(\mathcal{C}, J)/X$ the sheafified representables are identified with the objects $(\mathbf{a}y(c) \to X)$.

We stress that the argument must not be phrased as "**a** preserves local state classifiers", which is false [2, Rem. 3.24]. Since Theorem 4.1 is proved without sites, (i) and (ii) are not needed for the main result; they are only needed if one wants the site-theoretic description as such.

8. Related work

Local state classifiers were introduced in [2], which establishes the bijection between hyperconnected quotients of $\mathcal{E}$ and internal filters of Ξ, computes Ξ for presheaf toposes, G-sets and graphs, and proves that a Grothendieck topos is localic exactly when its local state classifier is terminal. Slice toposes are not treated there.

Menni [5] works inside $\mathcal{E}/Y$ with the same structure, calling the family $\{\xi_X\}$ a lax cocone and defining a map f to be *non-singular* when the folding inequality of Lemma 2.5 is an equality; his Theorem 6.4 shows that the non-singular maps into Y form a topos $\mathcal{N}(Y)$ arising as a hyperconnected quotient of $\mathcal{E}/Y$. He also characterizes Ξ for reflexive graphs as a classifier of lightly dense monomorphisms. A formula for $\Xi_{\mathcal{E}/X}$ does not appear.

[3] computes the local state classifier of a hyperconnected quotient. The present note fills in the complementary, localic half of the hyperconnected–localic factorization, since $\mathcal{E}/X \to \mathcal{E}$ is étale and hence localic.

On the side of relative topos theory, Caramello and Osmond [1] construct the over-topos at a model and identify it as a bilimit in the bicategory of Grothendieck toposes; the type of question is adjacent, but no connection with local state classifiers is made there, and we have not established one.

We have not found the formula of Theorem 4.1 in the literature. A full-text search of arXiv for the phrase "local state classifier" returns exactly the three papers [2, 5, 3], none of which treats $\Xi_{\mathcal{E}/X}$; the citation graph of [2] is the same closed set together with two papers on counting quotient toposes. This is of course only evidence about the phrase, not about the construction: a colimit of the monomorphism-subcategory may well appear elsewhere under a different name.

9. Open questions

Question 9.1 (Menni's non-singular quotient)**.** By Corollary 5.1, $\top_{\mathcal{E}/X} = \langle \xi_X, \mathrm{id}_X \rangle \colon X \rightarrowtail D_X$ generates a principal internal filter of $\Xi_{\mathcal{E}/X}$. Under the parameterization of [2], does this filter correspond to the hyperconnected quotient $\mathcal{E}/X \to \mathcal{N}(X)$ of [5, Thm. 6.4] given by the non-singular maps? Since f is non-singular exactly when $\xi_A = \xi_X \circ f$, i.e. when λ_f lands in the top section, this is plausible; it has not been computed. A verification would provide a check on Theorem 4.1 that is independent of its proof.

Question 9.2 (Sheaf-theoretic description)**.** Settle the two checks (i) and (ii) of Remark 7.2, so that the site-theoretic description of $\Xi_{\mathcal{E}/X}$ over a site of definition of $\mathcal{E}$ becomes available.

Question 9.3 (Elementary toposes)**.** Theorem 4.1 is conditional on the existence of Ξ in $\mathcal{E}$. For which elementary toposes does Ξ exist? Theorem 4.1 shows at least that the class of such toposes is closed under slicing.

Question 9.4 (Beyond étale morphisms)**.** $\mathcal{E}/X \to \mathcal{E}$ is the étale, hence localic, half of a hyperconnected–localic factorization, and [3] treats the hyperconnected half. Is there a formula for $\Xi_{\mathcal{F}}$ along an arbitrary localic geometric morphism $\mathcal{F} \to \mathcal{E}$, specializing to D_X for the étale case? More generally, is Ξ part of a fibred structure over the bicategory of $\mathcal{E}$-toposes, in the sense of [1]?

References

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