AN INTERNAL HOM FOR CONWAY ADDITION

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ABSTRACT. In *Games as recursive coalgebras* the category **Game** of impartial combinatorial games is shown to be symmetric monoidal closed for the Conway addition, but only by an abstract adjoint functor argument; Question 5.7 of that paper (Problem 5.0.5 of the author's problem list) asks what the internal hom actually is. We answer this. For games G, H we exhibit $[G, H]_+$ as the largest subgame $A(G, H)$ of the cofree game on the set $\mathbf{Set}(UG, UH)$ on which evaluation of labels is a game morphism. Concretely, a position of $[G, H]_+$ is a finite well-founded tree whose nodes are labelled by functions $UG \to UH$, subject to a hereditary option equation, and a move is the passage to a child. We prove that evaluation is a game morphism, that $A(G, H)$ is maximal with this property, that currying is a bijection natural in all three variables, and we compute several examples. In particular $[*1, *n]_+ \cong \mathbf{1}$ for every $n \geq 1$ and $[*2, *1]_+ = \emptyset$, so this internal hom is emphatically not the $-G + H$ of Joyal's compact closed category of games and strategies. The relation between the two remains open.

1. INTRODUCTION

Impartial combinatorial games can be organised into a category **Game** whose objects are games in the sense of Conway — a set of positions together with a finitely branching, well-founded option relation — and whose morphisms are maps that both preserve and lift options. Under the identification of games with recursive $\mathcal{P}_{\text{fin}}$-coalgebras [Hor25, Theorem 3.14], **Game** is a locally finitely presentable category [Hor25, Theorem B.9], comonadic over **Set** [Hor25, Theorem 3.15], and the Conway addition makes it symmetric monoidal closed [Hor25, Proposition 4.2].

The proof of closedness given there is indirect. By [Hor25, Corollary B.11], *every* monoidal structure on **Game** lifting the cartesian structure of **Set** is closed, because each $G + (-)$ is cocontinuous and the adjoint functor theorem for locally presentable categories applies. This produces a right adjoint but says nothing about what its positions and moves are. Accordingly:

Question 1.1 ([Hor25, Question 5.7]; Problem 5.0.5 of [HorP])**.** What is the internal hom with respect to the Conway addition?

This note answers Question 1.1 by an explicit construction. Write UG for the set of positions of a game G and θ_G for its option map. Let $S = \mathbf{Set}(UG, UH)$ and let $R(S)$ be the cofree game on S, whose positions are the S-labelled hereditarily finite sets [Hor25, Proposition B.14]. Call a position u of $R(S)$ *compatible* if for every descendant w of u (including u itself) and every $g \in UG$,

$$\theta_H\big(a_w(g)\big) \;=\; \big\{\, a_w(g') \;\big|\; g' \in \theta_G(g) \,\big\} \;\cup\; \big\{\, a_v(g) \;\big|\; v \in \theta_R(w) \,\big\}, \tag{E}$$

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where $a_w\colon UG \to UH$ is the label of w. Let $A(G,H) \subseteq R(S)$ be the set of compatible positions.

Theorem 1.2 (Theorem 4.1 below)**.** *$A(G,H)$ is a subgame of $R(S)$, the label evaluation $\mathrm{ev}(g,u) = a_u(g)$ is a game morphism $G + A(G,H) \to H$, and for every game K currying is a bijection*

$$\mathbf{Game}(G+K,\ H) \xrightarrow{\cong} \mathbf{Game}\big(K,\ A(G,H)\big),$$

natural in G, K and H. Hence $A(G,H)$ is the internal hom $[G,H]_+$ for the Conway addition.

Equation (E) has a direct game-theoretic reading, recorded in Section 5: a position of $[G,H]_+$ is a *plan*, that is, a map of option graphs $a\colon G \to H$, together with the finite set of plans reachable in one move, such that the options of $a(g)$ in H are *exactly* accounted for — with no repetition of accounting and no omission — by the moves of G at g and by the moves of the plan itself. Playing in $[G,H]_+$ is revising the plan.

Section 6 computes examples. The unit law $[\mathbf{1},H]_+ \cong H$ holds; $[G,\mathbf{1}]_+$ is $\mathbf{1}$ or $\emptyset$ according to whether G is discrete; and for the one-heap Nim games $*n$ we find $[*1,*n]_+ \cong \mathbf{1}$ for all $n \geq 1$ and $[*2,*1]_+ = \emptyset$. These already show that the present internal hom is not a categorification of Nim arithmetic and, in particular, does not agree with the $-G+H$ of the classical strategy categories of Joyal [Joy77] and Honsell–Lenisa–Redamalla [HLR12], where the morphism structure is by winning strategies rather than by exact coalgebra maps.

What is and is not claimed. Everything stated as a lemma, proposition or theorem below is proved here from the definitions; the only external inputs are the definitions and the description of the cofree game from [Hor25], and even the universal property of the latter is reproved in Lemma 2.6 so that the argument is self-contained. What is *not* claimed: (i) any relation between $[G,H]_+$ and player-switching duality, negation, or copycat strategies — the present setting does not contain the data these require, and Section 8 records what remains open; (ii) that the same maximal-compatible-subobject construction works for an arbitrary bifunctor lifting the cartesian product of **Set**, which would answer [Hor25, Questions 5.5, 5.6]; (iii) novelty. Regarding (iii): we have not been able to locate the construction below in the literature, but carving a mapping object out of a cofree object by imposing a structure-preservation condition is a general and old technique, and a negative search is not a proof of novelty.

2. Preliminaries

We follow [Hor25]. All the material in this section is from there, except Lemma 2.6, whose statement is [Hor25, Proposition B.14] but whose proof we give in the form we need.

Definition 2.1 ([Hor25, Definition 2.1])**.** A *game* G is a set UG together with a map $\theta_G\colon UG \to \mathcal{P}_{\mathrm{fin}}(UG)$ (equivalently a relation $\to_G$ with $\theta_G(g) = \{g' \mid g \to_G g'\}$) such that every option set is finite and there is no infinite path $g_0 \to_G g_1 \to_G \cdots$.

Finite branching plus the absence of infinite paths gives, by König's lemma, a *rank* $\rho_G(g) \in \mathbb{N}$: the length of the longest path starting at g. We use induction on ρ_G freely.

Definition 2.2 ([Hor25, Definition 3.12, Notation 3.13])**.** A *game morphism* $f\colon G \to H$ is a function $f\colon UG \to UH$ that is a graph morphism and lifts paths; equivalently, one for which

$$f\big[\theta_G(g)\big] = \theta_H\big(f(g)\big) \qquad (g \in UG). \tag{M}$$

Games and game morphisms form the category **Game**, and $U\colon \mathbf{Game} \to \mathbf{Set}$ is the forgetful functor.

We stress that (M) is an *equality* of sets, not an inclusion: **Game** is the category of recursive $\mathcal{P}_{\text{fin}}$-coalgebras and coalgebra homomorphisms [Hor25, Theorem 3.14], and this exactness is what makes the computations of Section 6 so rigid.

Definition 2.3 ([Hor25, Definition 2.10])**.** The *Conway addition* $G + K$ has $U(G+K) = UG \times UK$ and

$$\theta_{G+K}(g,k) = \big(\theta_G(g) \times \{k\}\big) \cup \big(\{g\} \times \theta_K(k)\big).$$

Ranks add, so $G+K$ is again a game. For morphisms $p\colon G \to G'$ and $q\colon K \to K'$ the map $(p+q)(g,k) = (p(g), q(k))$ satisfies (M), because $(p+q)[\theta_{G+K}(g,k)] = p[\theta_G(g)] \times \{q(k)\} \cup \{p(g)\} \times q[\theta_K(k)] = \theta_{G'+K'}(p(g), q(k))$; thus $+$ is a bifunctor. Its unit [Hor25, Proposition 4.2] is the game $\mathbf{1}$ with one position $\star$ and $\theta(\star) = \emptyset$: indeed $\theta_{\mathbf{1}+K}(\star, k) = \{\star\} \times \theta_K(k)$, so the projection $\mathbf{1} + K \to K$ is an isomorphism of games.

Definition 2.4 ([Hor25, Definition B.13])**.** For a set Λ, a Λ*-labelled hereditarily finite set* is a pair $w = (T_w, a_w)$ with T_w a finite set of Λ-labelled hereditarily finite sets and $a_w \in \Lambda$. The set of all of them is $\mathbb{H}_\Lambda$.

Definition 2.5 ([Hor25, Proposition B.14])**.** The *cofree game* on Λ is $R(\Lambda) = (\mathbb{H}_\Lambda, \theta_R)$ with $\theta_R(w) = T_w$, equipped with the counit $\epsilon_\Lambda\colon \mathbb{H}_\Lambda \to \Lambda$, $\epsilon_\Lambda(w) = a_w$.

Since children have smaller set-theoretic rank, $R(\Lambda)$ is a game. Note that $\mathbb{H}_\Lambda = \bigcup_n \mathbb{H}_{\Lambda,n}$ with $\mathbb{H}_{\Lambda,0} = \emptyset$ and $\mathbb{H}_{\Lambda,n+1} = \mathcal{P}_{\text{fin}}(\mathbb{H}_{\Lambda,n}) \times \Lambda$ is a genuine set, so no size issue arises below.

Lemma 2.6 (cofree transpose; [Hor25, Proposition B.14])**.** *For every game K and every function $\sigma\colon UK \to \Lambda$ there is a unique game morphism $\sigma^\flat\colon K \to R(\Lambda)$ with $\epsilon_\Lambda \circ U\sigma^\flat = \sigma$. It is given by $\sigma^\flat(k) = \big(\{\sigma^\flat(k') \mid k' \in \theta_K(k)\},\, \sigma(k)\big)$.*

Proof. Existence. Define $\sigma^\flat(k)$ by induction on the rank $\rho_K(k)$ using the displayed formula; this is legitimate since each $k' \in \theta_K(k)$ has strictly smaller rank, and the child set is finite because $\theta_K(k)$ is. By construction $\theta_R(\sigma^\flat(k)) = \sigma^\flat[\theta_K(k)]$, which is (M), and $\epsilon_\Lambda(\sigma^\flat(k)) = \sigma(k)$.

Uniqueness. Let $h\colon K \to R(\Lambda)$ satisfy $\epsilon_\Lambda \circ h = \sigma$. Then $h(k)$ has label $\sigma(k)$, and by (M) its child set is $h[\theta_K(k)]$. Induction on $\rho_K(k)$ gives $h(k) = \sigma^\flat(k)$. □

Finally, a *subgame* of a game X is a subset $B \subseteq UX$ closed under θ_X, regarded as a game by restricting θ_X; the inclusion is then a game morphism. We write $\operatorname{desc}(u)$ for the least set containing u and closed under θ.

Lemma 2.7 (image factorisation)**.** *Let $h\colon K \to X$ be a game morphism and $I = h[UK]$. Then I is a subgame of X, the corestriction $K \to I$ is a game morphism and is surjective on positions, and h factors as $K \twoheadrightarrow I \hookrightarrow X$.*

Proof. For $y = h(k) \in I$ we have $\theta_X(y) = h[\theta_K(k)] \subseteq I$, so I is closed under options; the same equality is (M) for the corestriction. □

3. The maximal compatible subgame

Fix games G and H and put $S = \mathbf{Set}(UG, UH)$. We read the label of a position w of $R(S)$ as a function $a_w\colon UG \to UH$.

Definition 3.1. $A(G,H) \subseteq \mathbb{H}_S$ is the set of u such that (E) holds for every $w \in \operatorname{desc}(u)$ and every $g \in UG$.

Lemma 3.2. *$A(G,H)$ is a subgame of $R(S)$.*

Proof. If $u \in A(G,H)$ and $v \in \theta_R(u)$ then $\mathrm{desc}(v) \subseteq \mathrm{desc}(u)$, so (E) holds throughout $\mathrm{desc}(v)$; hence $v \in A(G,H)$. □

Proposition 3.3 (evaluation)**.** *The function* $\mathrm{ev}_{G,H} \colon UG \times A(G,H) \to UH$, $\mathrm{ev}_{G,H}(g,u) = a_u(g)$, *is a game morphism* $G + A(G,H) \to H$.

Proof. Using Lemma 3.2 to identify $\theta_A(u)$ with $\theta_R(u)$,

$$\mathrm{ev}_{G,H}\big[\theta_{G+A}(g,u)\big] = \{a_u(g') \mid g' \in \theta_G(g)\} \cup \{a_v(g) \mid v \in \theta_R(u)\} = \theta_H\big(a_u(g)\big)$$

by (E) at u, which is (M) for $\mathrm{ev}_{G,H}$. □

Proposition 3.4 (maximality)**.** *Let* $B \subseteq \mathbb{H}_S$ *be a subgame of* $R(S)$ *such that* $(g,u) \mapsto a_u(g)$ *is a game morphism* $G + B \to H$. *Then* $B \subseteq A(G,H)$.

Proof. Let $u \in B$. Writing out (M) for this morphism at (g,u) and using $\theta_B(u) = \theta_R(u)$ gives exactly (E) at u, for every $g \in UG$. Since B is closed under options, $\mathrm{desc}(u) \subseteq B$ and the same holds at every $w \in \mathrm{desc}(u)$. Hence $u \in A(G,H)$. □

Thus $A(G,H)$ admits a description independent of Definition 3.1: it is the largest subgame of $R(\mathbf{Set}(UG,UH))$ on which label evaluation is a game morphism out of $G + (-)$.

4. The adjunction

Theorem 4.1. *For all games* G, K, H *the assignment*

$$\lambda \colon \mathbf{Game}(G+K,H) \longrightarrow \mathbf{Game}\big(K, A(G,H)\big), \qquad \lambda f = \big(\sigma_f\big)^\flat, \quad \sigma_f(k)(g) = f(g,k),$$

is a well-defined bijection, with inverse $h \mapsto \mathrm{ev}_{G,H} \circ (\mathrm{id}_G + h)$. *Consequently* $G + (-)$ *is left adjoint to* $A(G,-)$, *and* $A(G,H) = [G,H]_+$.

Proof. *λ lands in $A(G,H)$.* Let $f \colon G + K \to H$. Condition (M) for f at (g,k) reads

$$\theta_H\big(f(g,k)\big) = \{f(g',k) \mid g' \in \theta_G(g)\} \cup \{f(g,k') \mid k' \in \theta_K(k)\}. \tag{F}$$

Let $h_f = (\sigma_f)^\flat \colon K \to R(S)$ be the transpose of Lemma 2.6 and put $u = h_f(k)$. By that lemma $a_u = \sigma_f(k)$, i.e. $a_u(g) = f(g,k)$, and $\theta_R(u) = \{h_f(k') \mid k' \in \theta_K(k)\}$ with $a_{h_f(k')}(g) = f(g,k')$. Substituting, (F) *is* (E) at u. Now let $I = h_f[UK]$; by Lemma 2.7 it is a subgame of $R(S)$, and every element of I is of the form $h_f(k')$, so (E) holds at every element of I. Hence label evaluation is a game morphism $G + I \to H$ by the computation of Proposition 3.3, and Proposition 3.4 yields $I \subseteq A(G,H)$. Therefore h_f corestricts to a game morphism $\lambda f \colon K \to A(G,H)$, uniquely so because $A(G,H) \hookrightarrow R(S)$ is injective on positions.

The inverse is well defined. For $h \colon K \to A(G,H)$ the composite $\mathrm{ev}_{G,H} \circ (\mathrm{id}_G + h)$ is a game morphism $G + K \to H$ by Proposition 3.3 and bifunctoriality of $+$; on positions it is $(g,k) \mapsto a_{h(k)}(g)$.

Round trip from f. The label of $\lambda f(k)$ is $\sigma_f(k)$, so $\mathrm{ev}_{G,H}(\mathrm{id}_G + \lambda f)(g,k) = \sigma_f(k)(g) = f(g,k)$.

Round trip from h. Put $f = \mathrm{ev}_{G,H} \circ (\mathrm{id}_G + h)$, so $\sigma_f(k)(g) = a_{h(k)}(g)$, i.e. $\sigma_f(k) = a_{h(k)}$. Let $\iota \colon A(G,H) \hookrightarrow R(S)$ be the inclusion. Then $\iota \circ h \colon K \to R(S)$ is a game morphism whose composite with ϵ_S is $k \mapsto a_{h(k)} = \sigma_f$. By the uniqueness in Lemma 2.6, $\iota \circ h = (\sigma_f)^\flat = \iota \circ \lambda f$, and ι is injective on positions, so $h = \lambda f$. □

Proposition 4.2 (functoriality and naturality)**.** $A(-,-) \colon \mathbf{Game}^{\mathrm{op}} \times \mathbf{Game} \to \mathbf{Game}$ *is a functor, and the bijection of Theorem 4.1 is natural in* G, K *and* H.

Proof. Let $p\colon G' \to G$ and $r\colon H \to H'$, put $S' = \mathbf{Set}(UG', UH')$ and define $\Phi_{p,r}\colon S \to S'$ by $\Phi_{p,r}(a) = r \circ a \circ p$. The cofree functor gives $R(\Phi_{p,r})\colon R(S) \to R(S')$ by $R(\Phi)(T_w, a_w) = (\{R(\Phi)(v) \mid v \in T_w\}, \Phi(a_w))$; this satisfies (M) by construction, even though distinct children may be identified when Φ is not injective, since both sides of (M) are direct images.

Let $u \in A(G,H)$. The descendants of $R(\Phi)(u)$ are exactly the $R(\Phi)(w)$ with $w \in \mathrm{desc}(u)$. For such w and $g' \in UG'$, exactness of r, then (E) at w, then exactness of p give

$$\begin{aligned}\theta_{H'}\bigl(r(a_w(p(g'))\bigr) &= r\bigl[\theta_H(a_w(p(g')))\bigr]\\ &= \{r(a_w(g'')) \mid g'' \in \theta_G(p(g'))\} \cup \{r(a_v(p(g'))) \mid v \in \theta_R(w)\}\\ &= \{\Phi(a_w)(g_1) \mid g_1 \in \theta_{G'}(g')\} \cup \{\Phi(a_v)(g') \mid v \in \theta_R(w)\},\end{aligned}$$

which is (E) for $R(\Phi)(w)$. Hence $R(\Phi_{p,r})$ restricts to $A(p,r)\colon A(G,H) \to A(G',H')$. Since $\Phi_{\mathrm{id},\mathrm{id}} = \mathrm{id}$ and $\Phi_{p\circ p',\, r'\circ r} = \Phi_{p',r'} \circ \Phi_{p,r}$, functoriality follows from that of R together with the uniqueness of restrictions along the injections $A \hookrightarrow R$.

For naturality let also $q\colon K' \to K$ and $f\colon G + K \to H$, and set $f' = r \circ f \circ (p + q)$. Then $\sigma_{f'}(k')(g') = r\bigl(f(p(g'), q(k'))\bigr)$. On the other hand $\iota' \circ A(p,r) \circ \lambda f \circ q$ is a game morphism $K' \to R(S')$ whose label at k' is $\Phi_{p,r}(\sigma_f(q(k'))) = g' \mapsto r(f(p(g'), q(k')))$, the same function. By Lemma 2.6 the two morphisms coincide, whence

$$\lambda\bigl(r \circ f \circ (p+q)\bigr) = A(p,r) \circ \lambda(f) \circ q.$$

Naturality of the inverse follows, or may be checked directly on positions: $\mathrm{ev}(\mathrm{id} + A(p,r) \circ h \circ q)(g', k') = r\bigl(\mathrm{ev}(\mathrm{id} + h)(p(g'), q(k'))\bigr)$. □

Remark 4.3 (size and degenerate cases)**.** $S = \mathbf{Set}(UG, UH)$ is a set and $A(G,H) \subseteq \mathbb{H}_S$, so no class-sized objects occur. The extreme cases agree with the universal property. If $UG = \emptyset$ then S is a singleton and (E) is vacuous, so $A(G,H) = R(1) = \mathbb{H}$, the terminal game [Hor25, Proposition 3.31]; and indeed $G + K$ is then empty and $\mathbf{Game}(G+K, H)$ is a singleton. If $UG \neq \emptyset$ and $UH = \emptyset$ then $S = \emptyset$, so $A(G,H) = \emptyset$, the initial game; and $\mathbf{Game}(G+K, H)$ is a singleton or empty according as UK is empty or not, matching $\mathbf{Game}(K, \emptyset)$. Note also that $\mathbb{H} = R(1)$ is not a finite game, so the construction does not restrict to the full subcategory of games with finite position set; Theorem 4.1 is a statement about all of **Game**.

5. Reading $[G,H]_+$ as a game

Equation (E) is an equality of two sets, and splitting it into the two inclusions gives its meaning.

Lemma 5.1. *Let $u \in A(G,H)$ and $w \in \mathrm{desc}(u)$.*

(i) *$a_w\colon UG \to UH$ is a morphism of option graphs: if $g' \in \theta_G(g)$ then $a_w(g') \in \theta_H(a_w(g))$.*
(ii) *For every child $v \in \theta_R(w)$ and every $g \in UG$ we have $a_v(g) \in \theta_H(a_w(g))$: a move in $[G,H]_+$ is a move in H made simultaneously and uniformly at every position of G.*
(iii) *Every option of $a_w(g)$ in H arises in exactly one of these two ways.*

Proof. (i) and (ii) are the inclusion $\supseteq$ of (E) restricted to each of the two families; (iii) is the inclusion $\subseteq$. □

So a position of $[G,H]_+$ is a *plan* for translating G into H: a map a of option graphs, not required to be exact, together with the finite set of plans one may move to. Exactness of a is not required of a single plan; what is required is that the exactness defect of a at g — the options of $a(g)$ not of the form $a(g')$ — be precisely realised by the available revisions of the plan. Under currying, $\lambda f(k)$ is the plan $g \mapsto f(g,k)$, and its available revisions are the plans $g \mapsto f(g,k')$ for the options k' of k. Note that a plan with no revisions available is precisely an exact morphism $G \to H$; more generally, Lemma 5.1(iii) says that the moves of $[G,H]_+$ measure exactly how far each plan is from being a morphism.

Example 5.2 (the extra Nim token)**.** Take $G = *1$, the game with positions $\{1,0\}$, $\theta(1) = \{0\}$, $\theta(0) = \emptyset$; playing in $*1 + K$ is playing K with one extra token to spend. Write the label of a node w as the pair $(x_w, y_w) = (a_w(1), a_w(0))$. Then (E) becomes

$$\theta_H(y_w) = \{y_v \mid v \in \theta_R(w)\}, \qquad \theta_H(x_w) = \{y_w\} \cup \{x_v \mid v \in \theta_R(w)\}.$$

The first says the y-labelling is an exact morphism into H. The second forces $y_w \in \theta_H(x_w)$: spending the token must be a genuine move $x_w \to y_w$ of H.

6. Computations

Throughout, $*n$ denotes the one-heap Nim game with n tokens: positions $\{0,1,\dots,n\}$ with $\theta(i) = \{0,\dots,i-1\}$; thus $*0 = \mathbf{1}$. Call a game *discrete* if $\theta \equiv \emptyset$. All computations below were also checked by exhaustive machine enumeration of $A(G,H)$ and of both hom-sets for $K \in \{\emptyset, \mathbf{1}, *1, *2\}$ and the two-element discrete game.

Proposition 6.1 (unit law)**.** *$[\mathbf{1},H]_+ \cong H$, naturally in H.*

Proof. Here $S \cong UH$; write $h_w \in UH$ for the label of w. Condition (E) reduces to $\theta_H(h_w) = \{h_v \mid v \in \theta_R(w)\}$ at every descendant, i.e. to the statement that $w \mapsto h_w$ is exact. We claim $w \mapsto h_w$ is a bijection $A(\mathbf{1},H) \to UH$. *Injectivity* is by induction on the set-theoretic rank of w: the children of w are compatible of smaller rank, so by induction each child is determined by its label, and the set of children's labels is $\theta_H(h_w)$; hence the child set, and so w, is determined by h_w. *Surjectivity* is by induction on ρ_H: given h, put $\widehat{h} = (\{\widehat{h'} \mid h' \in \theta_H(h)\}, h)$. The bijection is a game morphism in both directions by construction. Naturality is Proposition 4.2. Alternatively, this is forced by $\mathbf{1} + K \cong K$ and Theorem 4.1. $\square$

Proposition 6.2. *$[G,\mathbf{1}]_+$ is the terminal game $\mathbb{H}$ if $UG = \emptyset$; the unit $\mathbf{1}$ if $UG \neq \emptyset$ and G is discrete; and the initial game $\emptyset$ otherwise.*

Proof. The first case is Remark 4.3. Otherwise S is a singleton, so $R(S) = \mathbb{H}$, and (E) at w for $g \in UG$ reads $\emptyset = \theta_{\mathbf{1}}(\star) = \{\star \mid \theta_G(g) \neq \emptyset\} \cup \{\star \mid \theta_R(w) \neq \emptyset\}$. If G is discrete this says exactly $\theta_R(w) = \emptyset$, so $A(G,\mathbf{1}) = \{\emptyset\} \cong \mathbf{1}$. If some g_0 has an option, the condition fails at every w, so $A(G,\mathbf{1}) = \emptyset$. $\square$

The second case is easy to confirm against the universal property: a morphism $X \to \mathbf{1}$ exists (and is then unique) iff X is discrete, and $G + K$ is discrete iff both G and K are.

Proposition 6.3. *$[*1,*n]_+ \cong \mathbf{1}$ for every $n \geq 1$, and $[*1,*0]_+ = [*2,*1]_+ = \emptyset$.*

Proof. For $[*1,*n]_+$ use the two equations of Example 5.2 with $H = *n$.

If $y_w = 0$ then $\theta_H(y_w) = \emptyset$, so w has no children, and the second equation gives $\theta_H(x_w) = \{0\}$, forcing $x_w = 1$. So the only node with $y_w = 0$ is the leaf ℓ with label $(1,0)$.

If $y_w = 1$ then all children have y-label 0, hence all children equal ℓ; as children form a set, w has exactly one child, namely ℓ, whose x-label is 1. The second equation gives $\theta_H(x_w) = \{1\} \cup \{1\} = \{1\}$. But in $*n$ every non-terminal position has 0 among its options and the terminal position has none, so no position has option set $\{1\}$: contradiction. Hence no node has $y_w = 1$.

If $y_w = j \geq 2$ then the children's y-labels are $\{0,\dots,j-1\} \ni 1$, so some child has y-label 1: impossible by the previous paragraph.

Therefore $A(*1,*n) = \{\ell\} \cong \mathbf{1}$.

$[*1, *0]_+ = \emptyset$ is Proposition 6.2. For $[*2, *1]_+$ write $a_w = (a_w(2), a_w(1), a_w(0))$ with values in $\{1, 0\}$. Equation (E) at $g = 1$ reads $\theta_H(a_w(1)) = \{a_w(0)\} \cup \{a_v(1) \mid v \in \theta_R(w)\}$; the right-hand side is non-empty, so $a_w(1) = 1$ and $\theta_H(1) = \{0\}$, whence $a_w(0) = 0$ and $a_v(1) = 0$ for all children v. But we have just shown $a_v(1) = 1$ for *every* node v; so w has no children. Then (E) at $g = 2$ reads $\theta_H(a_w(2)) = \{a_w(1), a_w(0)\} = \{1, 0\}$, and no position of $*1$ has option set $\{0, 1\}$. So no node is compatible. □

Remark 6.4. In Joyal's compact closed category of games and winning strategies [Joy77], and in its coalgebraic reworking [HLR12], the object classifying maps out of $G + (-)$ is $-G + H$, which for impartial games is $G + H$; for one-heap Nim this is $*n + *m$, equivalent to $*(n \oplus m)$ by the Sprague–Grundy theorem [Hor25, Theorem 2.17]. Proposition 6.3 gives $[*1, *2]_+ \cong \mathbf{1}$ where the Nim-arithmetic answer would be $*3$, and $[*2, *1]_+ = \emptyset$ where it would be $*3$ again. The internal hom of Theorem 4.1 is therefore *not* a categorification of Nim arithmetic, and the collapse is not an artefact of small examples: it comes from the exactness (M) of morphisms in **Game**, which is far more rigid than the existence of a winning strategy. It is worth noting that $[-, -]_+$ can nevertheless be large and non-degenerate — by Proposition 6.1 it is H when $G = \mathbf{1}$, and by Remark 4.3 it is the terminal game when G is empty.

Example 6.5. A non-degenerate case with $G = *1$. Let H have positions $\{d, c, b, e\}$ with $\theta(d) = \{c, b\}$, $\theta(c) = \theta(b) = \{e\}$, $\theta(e) = \emptyset$. Using Example 5.2: the nodes with $y = e$ are the two leaves $\ell_c = (c, e)$ and $\ell_b = (b, e)$ (a leaf needs $\theta_H(x) = \{y\}$, and both c and b qualify). A node with $y = c$ has children among the leaves with $\theta_H(x_w) = \{c\} \cup \{x\text{-labels}\}$; since $\theta_H(d) = \{c, b\}$ and no position has option set $\{c\}$, the child set must contain ℓ_b, giving two nodes $(\,(d, c), \{\ell_b\}\,)$ and $(\,(d, c), \{\ell_c, \ell_b\}\,)$. Symmetrically $y = b$ gives $(\,(d, b), \{\ell_c\}\,)$ and $(\,(d, b), \{\ell_c, \ell_b\}\,)$. A node with $y = d$ would need $\theta_H(x_w) = \{d\}$, impossible. So $[*1, H]_+$ has six positions, two of them terminal, and the four non-terminal ones have one or two options. The example shows that a position of $[G, H]_+$ genuinely remembers more than its label: $(\,(d, c), \{\ell_b\}\,)$ and $(\,(d, c), \{\ell_c, \ell_b\}\,)$ carry the same plan (d, c) but differ in which revisions are on offer.

7. Related work

The question answered here is [Hor25, Question 5.7], restated as Problem 5.0.5 of [HorP]; that paper supplies the category **Game**, the Conway addition, the abstract closedness [Hor25, Proposition 4.2, Corollary B.11] and the cofree game [Hor25, Proposition B.14] on which our construction rests, but no explicit internal hom.

The classical closed structure on games is of a different nature. Joyal [Joy77] takes the opposite $-G$ of a partisan game, exchanging the roles of Left and Right, defines a morphism $G \to H$ to be a winning strategy in $H - G$, and obtains a compact closed category in which the internal hom is $-G + H$. Honsell, Lenisa and Redamalla [HLR12] redo this coalgebraically, allowing non-well-founded games as elements of a final coalgebra and defining sum, negation and linear implication by final morphisms; their well-founded full subcategory recovers Joyal's. In both cases the objects are single games and the morphisms are strategies. In **Game** the objects are whole position sets with an option map and the morphisms are exact coalgebra homomorphisms, so neither the negation nor the copycat strategy is available, and Remark 6.4 shows the two internal homs genuinely disagree.

Kapulkin and Kershaw [KK24] classify the closed symmetric monoidal structures on the category of graphs and give explicit internal homs for the box product. That is a useful methodological comparison — an internal hom described by combinatorial data on vertices and edges — but their morphisms are graph morphisms rather than exact ones, so the formulas do not transfer. Altman and Lipparini [AL26] construct a ring structure on equivalence classes of partizan games; this concerns algebra on a quotient, not a right adjoint in a category of games, and is a different problem. Finally, Anel and Joyal [AJ13] carve mapping objects out of cofree coalgebras in the setting of Sweedler theory; the general shape of the technique — impose a compatibility condition inside a cofree object — is the same as ours, and we record this as the nearest general antecedent we are aware of, without claiming a formal implication in either direction.

8. Open questions

(1) **Player switching.** [Hor25] remarks after Question 5.7 that the closed structure "might be better understood when we consider the partisan games $\mathbf{RecCoalg}_{\mathcal{P}_{\mathrm{fin}}\times\mathcal{P}_{\mathrm{fin}}}$", and Problem 5.0.2 of [HorP] asks for a double category whose vertical part is **Game** and whose horizontal part is Joyal's category of games and strategies. Nothing in this note bears on that. Our $[G,H]_+$ is a vertical internal hom, and Remark 6.4 shows it is not $-G+H$; we do not know whether it arises as some vertical shadow of the horizontal structure, nor whether the plans of Lemma 5.1 admit a strategic reading.

(2) **Other liftings.** The proof of Theorem 4.1 uses the Conway option formula directly, through (F). Does the maximal compatible subgame construction work for an arbitrary bifunctor lifting the cartesian product of **Set**, e.g. the selective and conjunctive sums [Hor25, Examples 4.3, 4.4]? This would bear on [Hor25, Questions 5.5, 5.6]. One would need to redo the closure, maximality and naturality arguments; we have not done so.

(3) **When is $[G,H]_+$ non-degenerate?** Proposition 6.3 and Example 6.5 suggest that $[G,H]_+$ is empty or trivial unless H is "wide" relative to G — in Example 6.5 what made the difference was that H contained two distinct positions with the same option set. Is there a clean criterion on G,H for $[G,H]_+\neq\emptyset$? Equivalently, when does there exist an exact morphism $G\to H$ after adding enough spare moves?

(4) **Reduction of positions.** A position of $[G,H]_+$ is a labelled tree, which is a large amount of data; Example 6.5 shows the tree is not determined by its root label. Is there a canonical quotient or normal form — say, a bound on the depth that matters, or a presentation by a subset of UH^{UG} with extra structure?

(5) **Novelty.** We have not located this construction in the literature (see Section 7), but we have not carried out a systematic search of the coalgebraic mapping-object literature and do not claim priority.

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