QUIVER REPRESENTATION AND TOPOS THEORY

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Abstract.

Contents

In the initial section, we delve into basic Linear Algebra and its associated classifying toposes, which will serve as the foundation for subsequent applications. The subsequent section will concentrate on quiver representations.

1. Basic Linear Algebra and Classifying Toposes

In a formal sense, representation theory can be seen as a study of internal vector spaces (or more generally, modules) in various toposes of different nature, some of which are more algebraic, e.g. the presheaf toposes over (the free category) of a quiver, some of which are more topological, e.g. the topos of continuous G-actions, or the topos of light condensed sets, and many others.

In this section, we collect some basic logical theories in Linear Algebra and their classifying toposes. They will serve as base toposes of our study of representation theory.

The foundation of our study lies in the algebraic theory of vector spaces over a fixed field. Algebraic theories are of presheaf type, meaning their classifying toposes are simply presheaf toposes. Due to the ambiguity of natural language, Linear Algebra is never limited to the study of vector spaces, and soon we will go beyond algebraic theories, e.g. the coherent theory of discrete/geometric fields. Within both contexts, as outlined by Olivia Caramello's methodology 'Toposes as Bridges', classifying toposes can manifest itself in diverse presentations. We will endeavor to explore as many presentations as possible. In the subsequent stages, from a theoretical standpoint, these presentations may offer greater flexibility in comprehending more complex results. Furthermore, from a practical perspective, they could simplify the computations around toposes when a specific presentation is chosen.

2020 *Mathematics Subject Classification.* MSC.
Key words and phrases. Keywords.
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1.1. Vector Spaces and Modules.

The theory of k-vector spaces $\mathbb{V}\mathrm{ect}_k$ over the signature $\Sigma = \left\{\mathsf{V}; +, -, (\tau_a)_{a\in k}, \mathsf{0}\right\}$ is algebraic.

The classifying topos is equivalent to any of the following toposes (detailes to be added later):

- $\mathbf{Sh}\left(\mathcal{C}_{\mathbb{V}\mathrm{ect}_k}, J_{\mathbb{V}\mathrm{ect}_k}\right)$, where $\left(\mathcal{C}_{\mathbb{V}\mathrm{ect}_k}, J_{\mathbb{V}\mathrm{ect}_k}\right)$ is the geometric syntactic site of $\mathbb{V}\mathrm{ect}_k$;
- $\mathbf{PSh}\left(\mathcal{C}^{\mathrm{cart}}_{\mathbb{V}\mathrm{ect}_k}\right)$, where $\mathcal{C}^{\mathrm{cart}}_{\mathbb{V}\mathrm{ect}_k}$ is the cartesian syntactic category of $\mathbb{V}\mathrm{ect}_k$;
- $[\mathbf{Vect}^{\omega}_k, \mathbf{Set}]$, where $\mathbf{Vect}^{\omega}_k$ is the category of finitely presentable k-vector spaces and linear maps;
- $[\mathcal{M}at_k, \mathbf{Set}]$, where $\mathcal{M}at_k$ the category of matrices with natural numbers as objects and matrices as morphisms;
- $\mathbf{PSh}\left(\mathbf{FinSet} \rtimes \mathbb{P}_{\mathbf{Vect}_k}\right)$, where $\mathbb{P}_{\mathbb{V}\mathrm{ect}_k} : \mathbf{FinSet}^{\mathrm{op}} \to \wedge\text{-}\mathbf{SLat}$ the primary doctrine of $\mathbb{V}\mathrm{ect}_k$
- $\mathbf{Sh}\left(\left(\mathbf{Vect}^{\omega}_k \downarrow k^{(-)}\right)^{\mathrm{op}}, J^{\mathrm{triv}}_{k^{(-)}}\right)$, where

$$k^{(-)} : \mathcal{M}at_k \to \mathbf{Vect}^{\omega}_k : (n \xrightarrow{M} m) \mapsto \left(k^n \xrightarrow{M} k^m\right)$$

and $J^{\mathrm{triv}}_{k^{(-)}}$ is the induced Giraud topology of J^{triv}; this presentation is an easy application of Caramello's canonical fibred site with respect to the equivalence of toposes induced by the functor $k^{(-)}$;

- $\mathbf{Sh}\left(\left(\mathbf{Vect}^{\omega}_k \downarrow^{*} k^{(-)}\right)^{\mathrm{op}}, J^{\mathrm{triv}}_{k^{(-)}}\right)$, where $\mathbf{Vect}^{\omega}_k \downarrow^{*} k^{(-)}$ is the subcategory of the comma category $\mathbf{Vect}^{\omega}_k \downarrow k^{(-)}$ whose objects are only monomorphisms $V \to k^n$ and $J^{\mathrm{triv}}_{k^{(-)}}$ is the induced Giraud topology of J^{triv}; this presentation is an easy application of Caramello's reduceced canonical fibred site with respect to the equivalence of toposes induced by the functor $k^{(-)}$;
- $\left[\mathbf{Vect}^{\omega}_k \downarrow^{\cong} k^{(-)}, \mathbf{Set}\right]$, where $\mathbf{Vect}^{\omega}_k \downarrow^{\cong} k^{(-)}$ is is the subcategory of the comma category $\mathbf{Vect}^{\omega}_k \downarrow k^{(-)}$ whose objects are only isomorphisms.

Remark 1.1. Since it is not constructively provable that every finitely presentable vector space is free,

1.2. Geometric Fields.

1.3. Quadratic Forms and Symmetric Bilineat Forms.

2. Quiver Representation and Classifying Toposes

2.1. Quivers and Graphs.

2.2. Gabriel's Theorem.