LOCAL STATE CLASSIFIER CHARACTERIZES THE AXIOM OF CHOICE

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Abstract.

Contents

1. Preliminaries on Hyperconnected quotients and local state classifier

memo: This section is a copoy of [Hor25, Section 2] This section is a 2-page summary of the paper [Hor24], which defines and studies the notion of a local state classifier.

1.1. Hyperconnected quotients.

1.1. Hyperconnected quotients. This subsection aims to recall the preliminaries on hyperconnected geometric morphisms. See [Joh81] or [Joh02, A.4.6] for more details.

Definition 1.1 (Hyperconnected geometric morphisms)**.** A geometric morphism $f\colon \mathcal{E} \to \mathcal{F}$ is said to be **hyperconnected** if it is connected (i.e. $f^*\colon \mathcal{F} \to \mathcal{E}$ is fully faithful) and its counit $\epsilon_X\colon f^*f_* \to \mathrm{id}_{\mathcal{E}}$ is monic.

In this paper, a **hyperconnected quotiet** of a topos $\mathcal{E}$ means (an equivalence class of) a hyperconnected geometric morphism from $\mathcal{E}$. Since f^* is fully faithful for a hyperconnected quotient $f\colon \mathcal{E} \to \mathcal{F}$, we can regard $\mathcal{F}$ as a (replete) full subcategory of $\mathcal{E}$. With this identification, we will write '$X \in \mathrm{ob}(\mathcal{E})$ belongs to $\mathcal{F}$' for '$X \in \mathrm{ob}(\mathcal{E})$ belongs to the essential image of f^*' in this paper. This does not cause any serious problem since we will not distinguish between two mutually equivalent hyperconnected quotients.

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1.2. **Local state classifier.** In this subsection, we will briefly explain the notion of a local state classifier. For more proofs, informal explanations, and examples, see the original article [Hor24].

1.2.1. *Definition.*

Definition 1.2 ([Hor24, Definition 3.4]). The **local state classifier** of a category $\mathcal{E}$ is the colimit of all monomorphisms of $\mathcal{E}$, if it exists. In other words, it is an object Ξ equipped with a family of morphisms $\{\xi_X : X \to \Xi\}_{X \in \mathrm{ob}(\mathcal{E})}$, such that they form a colimit cocone under the faithful embedding functor $\mathcal{E}_{\mathrm{mono}} \rightarrowtail \mathcal{E}$.

The definition of a local state classifier is quite transcendental, and even a (small-)cocomplete category might not admit a local state classifier. However, we can prove the following proposition:

Proposition 1.3 ([Hor24, Section 3.16]). *Every Grothendieck topos $\mathcal{E}$ has a local state classifier.*

1.2.2. *Inducing full subcategories.* How is a local state classifier related to the classification of hyperconnected quotients? Since Ξ is just an object of $\mathcal{E}$ and a hyperconnected quotient is a (very nice) subcategory of $\mathcal{E}$, they might seem unrelated.

The answer is, in short, that we can construct a full subcategory of $\mathcal{E}$ from any subobject of Ξ. Let $\mathcal{E}$ be a category with a local state classifier Ξ. For any subobject $\iota_F : F \rightarrowtail \Xi$, we can define a full subcategory $\mathcal{E}_F \hookrightarrow \mathcal{E}$ by

$$X \in \mathrm{ob}(\mathcal{E}_F) \iff \begin{tikzcd} & F \arrow[d, tail, "\iota_F"] \\ X \arrow[ur, dashed, "\exists"] \arrow[r, "\xi_X"'] & \Xi. \end{tikzcd} \tag{1}$$

In other words, we define the full subcategory $\mathcal{E}_F$ of $\mathcal{E}$, specifying objects by

$$\mathrm{ob}(\mathcal{E}_F) \coloneqq \{X \in \mathrm{ob}(\mathcal{E}) \mid \text{ the morphism } \xi_X \text{ factors through } F \rightarrowtail \Xi\}.$$

In this paper, for each object $X \in \mathrm{ob}(\mathcal{E}_F)$, we write $\xi^F_X : X \to F$ for the unique lift of ξ_X along ι_F

$$\begin{tikzcd} & F \arrow[d, tail, "\iota_F"] \\ X \arrow[ur, "\xi^F_X"] \arrow[r, "\xi_X"'] & \Xi. \end{tikzcd}$$

1.2.3. *The order structure.* Although the definition of a local state classifier makes sense for any category, it behaves better in cartesian closed categories. First and foremost, in a cartesian closed category, the local state classifier acquires a canonical semilattice structure reflecting the cartesian structure of $\mathcal{C}$.

Proposition 1.4 ([Hor24, Proposition 3.27]). *If a cartesian closed category (in particular, an elementary topos) $\mathcal{E}$ admits a local state classifier $\{\xi_X : X \to \Xi\}_{X \in \mathrm{ob}(\mathcal{E})}$, there exists a unique internal $\wedge$-semilattice structure on Ξ such that the diagram*

$$\begin{tikzcd} & X_1 \times \cdots \times X_n \arrow[dl, "(\xi_{X_1}) \times \cdots \times (\xi_{X_n})"'] \arrow[dr, "\xi_{(X_1 \times \cdots \times X_n)}"] & \\ \Xi^n \arrow[rr, "\wedge"'] & & \Xi \end{tikzcd}$$

commutes for any finite sequence of objects $X_1, \dots, X_n \in \mathrm{ob}(\mathcal{E})$, $n \geq 0$.

This internal semilattice structure on Ξ induces a (usual) semilattice structure on each homset $\mathcal{E}(X, \Xi)$ for each object $X \in \mathrm{ob}(\mathcal{E})$. Therefore, each homset $\mathcal{E}(X, \Xi)$ admits a natural partial order defined by $f \leq g \iff f \wedge g = f$. A subobject $\iota_F : F \rightarrowtail \Xi$ is said to be an **internal filter**, if each subset $\mathcal{E}(X, F) \rightarrowtail \mathcal{E}(X, \Xi)$ is a filter in the usual sense (i.e., upward closed and closed under finite meets $\top, \wedge$). (In [Hor24], the author adopts a diagrammatic definition of an internal filter so that it makes sense even for locally large categories.)

1.2.4. *The classification theorem.* The paper [Hor24] proves that, if the category $\mathcal{E}$ is an elementary topos and the subobject $F \rightarrowtail \Xi$ is an internal filter, the induced full subcategory $\mathcal{E}_F$ is also an elementary topos, and the embedding $\mathcal{E}_F \hookrightarrow \mathcal{E}$ admits a right adjoint defining a hyperconnected geometric morphism $f_F\colon \mathcal{E} \to \mathcal{E}_F$. The main theorem of [Hor24] (Theorem 1.5) states that this construction $F \mapsto \mathcal{E}_F$ provides a bijective correspondence between the internal filters of Ξ and the hyperconnected quotients of $\mathcal{E}$.

Theorem 1.5 ([Hor24, Theorem 4.1][1])**.** *If an elementary topos $\mathcal{E}$ has a local state classifier Ξ, there exists a bijective correspondence between*

- *hyperconnected quotients of the topos $\mathcal{E}$, and*
- *internal filters of the local state classifier Ξ.*

The paper [Hor24] also provides the description of the corresponding lex comonad $\mathbb{G} \coloneqq f^*f_*\colon \mathcal{E} \to \mathcal{E}$ with its counit $\epsilon\colon \mathbb{G} \to \mathrm{id}_{\mathcal{E}}$.

$$\begin{tikzcd}
& \mathcal{E}_F \arrow[dr, "f^*"] \arrow[d, phantom, "\shortparallel"] & \\
\mathcal{E} \arrow[ur, "f_*"] \arrow[rr, "\mathbb{G}"] \arrow[rr, bend right=50, "\mathrm{id}_{\mathcal{E}}"'] \arrow[rr, phantom, bend right=25, "\Downarrow\epsilon"] & {} & \mathcal{E}
\end{tikzcd}$$

It states that the monic counit map $\epsilon_X\colon \mathbb{G}X \rightarrowtail X$ for each object $X \in \mathrm{ob}(\mathcal{E})$ is given by the pullback diagram

$$\begin{tikzcd}
\mathbb{G}X \arrow[r, "\xi^F_{\mathbb{G}X}"] \arrow[d, tail, "\epsilon_X"] \arrow[dr, phantom, very near start, "\lrcorner"] & F \arrow[d, tail, "\iota_F"] \\
X \arrow[r, "\xi_X"'] & \Xi.
\end{tikzcd} \tag{2}$$

2. LSC and set-theoretic models

2.1. Atom and atomic quotient.

2.1.1. *Atomic quotient.*

Definition 2.1. An atomic quotient of an elementary topos $\mathcal{E}$ is a quotient topos $f\colon \mathcal{E} \to \mathcal{F}$ where f^* is logical.

2.1.2. *The von Neumann hierarchy with atoms in a topos.* (cf. [Fre80])

Definition 2.2 (The von Neumann hierarchy)**.** For a Grothendieck topos $\mathcal{E}$ and an object A[2], we define V^A_α for every ordinal α inductively as follows:

- α **is zero:** $V^A_0 \coloneqq 0_{\mathcal{E}}$
- $\alpha = \beta + 1$**:** $V^A_{\beta+1} \coloneqq A + \mathcal{P}(V^A_\beta)$ memo: V^A_1 is the set of all atoms and the real empty set
- α**: limit:** $V^A_\alpha \coloneqq \mathrm{colim}_{\beta<\alpha} V^A_\beta$ (The diagram is inductively defined by the singleton embedding $\{\cdot\}_{V^A_\beta}\colon V^A_\beta \rightarrowtail \mathcal{P}(V^A_\beta) = V^A_{\beta+1}$)

An object X is said to be **well-founded over** A if there exist an ordinal α and a monomorphism $X \rightarrowtail V^A_\alpha$. The topos $\mathcal{E}$ is said to be **well-founded over** A if every object of $\mathcal{E}$ is well-founded over A.

For any ordinal $\beta < \alpha$, we define the β-th ordinal in $V_\alpha(A)$ by

[1] In [Hor24], another correspondant, internal semilattice homomorphisms $\Xi \to \Omega$, is given.

[2] Intended to be the set of atoms

Todo 2.3. *For any Grothendieck topos $\mathcal{E}$ and an object $A \in \mathrm{ob}(\mathcal{E})$, the class of well-founded objects over A is closed under*

1. *taking subobjects,*
2. *taking power objects,*
3. *taking finite products,*
4. *taking small coproducts,*
5. *taking quotient objects,*
6. *taking exponential objects.*

Proof. We will prove them one by one.

1. This immediately follows from the definition.
2. Let α be an ordinal, and let $f\colon X \rightarrowtail V_\alpha(A)$ be a monomorphism. Then, one can prove that the internal existential quantifier $\exists_f\colon \mathcal{P}(X) \rightarrowtail \mathcal{P}(V_\alpha(A)) = V_{\alpha+1}$ is monic.
3.

□

3. LSC and etendue

Conjecture 3.1. *A Grothendieck topos $\mathcal{E}$ is an étendue if and only if its local state classifier Ξ admits the minimum element $\bot\colon 1_{\mathcal{E}} \to \Xi$.*

Conjecture 3.2 (Subconjecture)**.** *A Grothendieck topos $\mathcal{E}$ satisfies the internal axiom of choice if and only if $\mathcal{E}$ is Boolean and its local state classifier Ξ admits the minimum element $\bot\colon 1_{\mathcal{E}} \to \Xi$.*

Due to the following fact, studying LSC of étendues is close to studying all Grothendieck topoi.

Fact 3.3 ([Ros82, Theorem 3.1])**.** *For every Grothendieck topos $\mathcal{F}$, there is an étendue $\mathcal{E}$ and a hyperconnected geometric morphism $\mathcal{E} \to \mathcal{F}$.*

memo: How is this Rosenthal's covering theorem rephrased by LSC?

The concept of topos as a "generalized locale" plays the role of a "space with rich self-automorphisms," or, in other words, a "folded space." In the order structure of LSC, a more "collapsed" state is considered larger, while a more "unfolded" state is considered smaller. For instance, in the topos of graphs, a loop edge is larger than a non-loop edge. In the topos of group actions, a trivial action on a point is larger than a free action.

So, when does LSC have a minimum (global) element? The slogan would be, "the existence of the unfolding," and this is none other than an étendue!

Conjecture 3.4. *For a Grothendieck topos $\mathcal{E}$, the following conditions are equivalent (?)*

- *its LSC Ξ has a bottom $\bot\colon 1 \to \Xi$*
- *$\mathcal{E}$ is étendue.*

Plan: This conjecture will be proven by rewriting [KM91] in terms of LSC.

Fact 3.5. *A Grothendieck topos $\mathcal{E}$ is*

- *étendue if and only if it has a site $(\mathcal{C}, J)$, where all morphisms of $\mathcal{C}$ are monic. [KM91]*
- *Boolean étendue iff $\mathcal{E}$ satisfies the internal axiom of choice.*

A presheaf topos $\mathbf{PSh}(\mathcal{C})$ is étendue if and only if all morphisms of $\mathcal{C}$ are monic. [Ros81]

Proposition 3.6. *Theorem 3.4 is true for*

- *localic topoi*

- *presheaf topoi*

Proof. For a localic topoi $\mathcal{E}$, it's trivial since $\mathcal{E}$ is étendue and its LSC is terminal. For a presheaf, the LSC Ξ is the presheaf of all quotient objects of the representables. By the concrete calculation, Ξ has a bottom, if and only if $Y(f)\colon Y(c) \to Y(d)$ is monic for every $f\colon c \to d$, which means every morphism in $\mathcal{C}$ is monic. $\square$

Example 3.7. $\mathbf{Cont}(\hat{\mathbb{Z}})$ is not étendue, since it does not satisfy the internal axiom of choice [Fre80; FS90]. Its lsc does not have the bottom, see theorem A.2.

3.1. **Rewriting Kock and Meordijk.** This subsection aims to rewrite [KM91] in terms of a local state classifier.

Definition 3.8 ([KM91])**.** For a geometric morphism $\gamma\colon \mathcal{E} \to \mathcal{S}$ between two elementary topoi, a morphism $f\colon A \to B$ is said to be **locally monic** relative to γ, if there exist the diagram

$$\begin{tikzcd} A' \arrow[r, tail, "f'"] \arrow[d, two heads, "q"] & \gamma^* I \times B \arrow[d, "\mathrm{proj}"] \\ A \arrow[r, "f"] & B, \end{tikzcd}$$

where f' is monic and q is epic.

For a Grothendieck topos $\mathcal{E}$, a morphism f in $\mathcal{E}$ is said to be locally monic if it is locally monic relative to the global section geometric morphism $\gamma\colon \mathcal{E} \to \mathbf{Set}$. For a morphism $f\colon A \to B$, f is locally monic if and only if there is a covering $\{U_\lambda \rightarrowtail A\}_{\lambda\in\Lambda}$ such that each restricted morphism

$$f|_{U_\lambda}\colon U_\lambda \rightarrowtail A \to B$$

is monic.

Conjecture 3.9. *A morphism $f\colon A \to B$ in a Grothendieck topos $\mathcal{E}$ is locally monic (relative to the global section geometric morphism) if and only if*

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\xi_A"'] & & B \arrow[dl, "\xi_B"] \\ & \Xi & \end{tikzcd}$$

commutes.

Colloquially, this conjecture states that 'locally monic' is equivalent to saying that it preserves local states without collapsing them. See theorem B.1. This conjecture holds for localic topoi and presheaf topoi.

Lemma 3.10. *If a morphism $f\colon A \to B$ in a Grothendieck topos $\mathcal{E}$ is locally monic, then*

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\xi_A"'] & & B \arrow[dl, "\xi_B"] \\ & \Xi & \end{tikzcd}$$

commutes.

Proof. There is a covering $\{U_\lambda \rightarrowtail A\}_{\lambda\in\Lambda}$ such that each restricted morphism

$$f|_{U_\lambda}\colon U_\lambda \rightarrowtail A \to B$$

is monic. Then, the outside square of

$$\begin{tikzcd} & U_\lambda \arrow[dl, tail] \arrow[dr, tail, "f|_{U_\lambda}"] & \\ A \arrow[rr, "f"] \arrow[dr, "\xi_A"'] & & B \arrow[dl, "\xi_B"] \\ & \Xi & \end{tikzcd}$$

commutes. Since $\{U_\lambda \rightarrowtail A\}_{\lambda\in\Lambda}$ is jointly epimorphic, this completes the proof. □

(For the converse question: relationship with the existence of reduced subobjects coverings)

3.2. **Torsion-free objects.**

Lemma 3.11. *For an object X of a topos $\mathcal{E}$ with a local state classifier Ξ, the following conditions are equivalent:*

- *$\pi\colon \Xi_X \to X$ is the terminal object of $\mathcal{E}/X$*
- *For any object Y and any map $f\colon Y \to X$, the composite map $\xi_X \circ f\colon Y \to \Xi$ is an bottom element in the $\wedge$-semilattice $\mathcal{E}(Y, \Xi)$*

Conjecture 3.12. *The above condition should be also equivalent to*

- *$\mathcal{E}/X$ is localic*
- *X is torsion-free in the sense of [KM91].*

3.3. **Inhabitedness.**

Lemma 3.13. *If an object X satisfies the condition in Theorem 3.11, then the map $\xi_X\colon X \to \Xi$ factors through the support of X:*

$$\begin{tikzcd} X \arrow[d, two heads] \arrow[dr, "\xi_X"] & \\ T \arrow[r, tail, dashed, "\exists"] \arrow[d, tail] & \Xi \\ 1_{\mathcal{E}} & \end{tikzcd}$$

Proof. Due to theorem 3.11, the diagram

$$X \times_\Xi X \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} X \xrightarrow{\xi_X} \Xi$$

commutes. The lemma follows since the support of X, denoted by T, is the coequalizer of this diagram, since a topos is regular. □

Lemma 3.14. *If there exists an inhabited object that satisfies the condition in Theorem 3.11, then Ξ has a global bottom element $\perp\colon 1_{\mathcal{E}} \to \Xi$.*

Assuming the next conjecture

Conjecture 3.15. *In any Grothendieck topos $\mathcal{E}$ (or its relativization), there is an object B such that $\xi_B\colon B \to \Xi$ is epic.*

which is closely related to Section C, we can construct an inhabited and torsion-free (in the sense of Theorem 3.11) object X by a pullback:

$$\begin{tikzcd} X \arrow[r, two heads, "!"] \arrow[d, tail, "\iota"] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, tail, "\perp"] \\ B \arrow[r, two heads, "\xi_B"] & \Xi, \end{tikzcd}$$

since we have

$$\begin{tikzcd} X \arrow[r, two heads, "!"] \arrow[d, tail, "\iota"] \arrow[dr, "\xi_X"] & 1 \arrow[d, tail, "\perp"] \\ B \arrow[r, two heads, "\xi_B"] & \Xi \end{tikzcd}$$

Appendix A. LSC of continuous action toposes

Corollary A.1. *For a topological group G, the LSC of $\mathbf{Cont}(G)$ is the set of open subgroups of G, equipped with the right conjugate actions.*

Example A.2 (LSC of loops)**.** The LSC of $\mathbf{Cont}(\hat{\mathbb{Z}})$ is the semilattice of positive integers with the (reversed) divisibility order and the trivial action.

Appendix B. LSC of slice topos memo: ongoing

Motivated by section 3, we will describe the LSC of the slice topos. Our starting point is the next lemma, which is proven in [Hor24].

Lemma B.1 ([Hor24])**.** *For any morphism $f\colon Y \to X$,*

$$\xi_Y \leq \xi_X \circ f.$$

This allows us to define

$$\begin{tikzcd} Y \arrow[dr, dashed, "\zeta_f"] \arrow[drr, bend left, "\langle \xi_Y, \xi_X \circ f \rangle"] \arrow[ddr, bend right, "f"'] & & \\ & P \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \leq_\Xi \arrow[d, two heads, "\pi_2"] \\ & X \arrow[r, "\xi_X"] & \Xi \end{tikzcd}$$

Definition B.2. For a topos $\mathcal{E}$ with a local state classifier Ξ and an object $X \in \mathrm{ob}(\mathcal{E})$, we define Ξ_X by

$$\Xi_X \coloneqq \{(s, x) \in \Xi \times X \mid s \leq \xi_X(x)\}$$

interpreted in the internal language in $\mathcal{E}$.

This is exactly the same as the pullback

$$\begin{tikzcd} \Xi_X \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \leq_\Xi \arrow[d, two heads, "\pi_2"] \\ X \arrow[r, "\xi_X"] & \Xi, \end{tikzcd}$$

Example B.3.

Conjecture B.4. *For a topos $\mathcal{E}$ with LSC Ξ, the LSC of the slice topos $\mathcal{E}/X$ is given by the pullback*

$$\begin{tikzcd} P \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \leq_\Xi \arrow[d, two heads, "\pi_2"] \\ X \arrow[r, "\xi_X"] & \Xi, \end{tikzcd}$$

with the cocone maps $\{\zeta_f \colon Y \to P\}_{f\colon Y\to X}$

Appendix C. Relationship with bounds memo: ongoing

There should be some connection with the notion of bound and LSC. The reasons why I think so include

- Every Grothendieck (**Set**-bounded) topos has a LSC, every finite presheaf (**FinSet**-bounded) topos (over finite category) has a LSC, but $\mathbf{FinSet}^{\mathbb{Z}}$, which is not bounded over **FinSet** doesn't.
- Informally speaking, an object B is a bound, if and only if "every state of every object is a quotient state of a state of B."

Conjecture C.1. *If a topos $\mathcal{E}$ has a local state classifier, then every bounded $\mathcal{E}$-topos has a local state classifier.*

Conjecture C.2. *An object X of a Grothendieck topos $\mathcal{E}$ is a bound, if and only if $\xi_X \colon X \to \Xi$ is downward unbounded, in the sense that every upward closed subobject of Ξ containing $\mathrm{Im}(\xi_X)$ is Ξ itself.*

Example C.3. Even if the hyperconnected quotient generated by B is $\mathcal{E}$ itself, $\mathcal{E}$ might not be a bound. For example, the object $B \coloneqq \mathbb{Z}/2\mathbb{Z} + \mathbb{Z}/3\mathbb{Z}$ in $\mathbf{PSh}(\mathbb{Z}/6\mathbb{Z})$ is not a bound, but every non-trivial hyperconnected quotient does not contain B.

So what we need to consider is the **broader correspondence** in [Hor24]. memo: And is related to Menni's paper [Men21].

Conjecture C.4 (memo: Proven, This is also proven by P.T. Johnstone. Its presheaf case is proven in [Men25])**.** *As a restriction of [Hor24, broader correspondence], we obtain a one-to-one correspondence between*

- *Upward closed subobject of Ξ, and*
- *Coreflective full subcategory closed under subquotients.*
- *Order-preserving map $\Xi \to \Omega$*

memo: It may subsume the monic skelta by Menni

Example C.5 (Galois theory)**.** memo: Check and generalize it Let $\mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p)$ be the absolute Galois group of a finite field $\mathbb{F}_p$. The algebraic closure equipped with the action $\mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p)$ is an internal ring of $\mathbf{Cont}(\mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p))$, with surjective $\xi_{\overline{\mathbb{F}_p}}$. For each open subgroup $S \subset \mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p)$ and its corresponding hyperconnected geometric morphism $\mathbf{Cont}(\mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p)) \to \mathbf{PSh}(\mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p)/S)$, the counit $K_S \rightarrowtail \overline{\mathbb{F}_p}$ is the embedding of the Galois-correspondant.

Appendix D. Internally complete semilattice

Definition D.1. For an internal poset P in a topos $\mathcal{E}$ and an object I, **the I-indexed meet** $\wedge_I$ is the internal right adjoint of the diagonal morphism

$$\Delta_I \colon \Xi \to \Xi^I.$$

memo: check Externally speaking, the $\wedge_I$ exists if and only if there exists a right adjoint to

$$- \circ \pi_1 : \mathcal{E}(X, \Xi) \to \mathcal{E}(X \times I, \Xi)$$

for any $X \in \mathrm{ob}(\mathcal{E})$ that is natural in X.

memo: Possibly, the right condition might be the internal completeness, not the special case of it, namely the existence of the bottom object.

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