TOPOI OF AUTOMATA II: HYPERCONNECTED GEOMETRIC MORPHISMS AND SYNTACTIC TOPOI

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Abstract. For a hyperconnected geometric morphism from the topos of Σ-sets, we formulate language recognition using the local state classifier. For a language class C, we construct the associated syntactic Grothendieck topos $\mathcal{T}(C)$.

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Contents

1. Introduction

As we have seen so far, the hyperconnected geometric morphism

$$\Sigma\text{-}\mathbf{Set} \xrightarrow{h} \Sigma\text{-}\mathbf{Set}_{\mathrm{o.f.}}$$

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plays a central role in our theory of regular languages. This work considers other hyperconnected geometric morphisms

$$\begin{tikzcd} & \mathcal{E} \\ \Sigma\text{-}\mathbf{Set} \arrow[ur] \arrow[r] \arrow[dr] & \mathcal{F} \\ & \mathcal{G} \end{tikzcd}$$

and their associated classes of languages.

2. Hyperconnected geometric morphisms from Σ-Set

2.1. **Hyperconnected geometric morphisms.** See [Joh02; Joh81] for the detailed definitions of hyperconnected geometric morphisms.

Definition 2.1 (Hyperconnected geometric morphisms)**.** A geometric morphism $f\colon \mathcal{E} \to \mathcal{F}$ is ***hyperconnected*** if f^* is fully faithful and f satisfies the following equivalent conditions:

- The essential image of f^* is closed under taking subquotients.
- The essential image of f^* is closed under taking subobjects.
- The essential image of f^* is closed under taking quotient objects.
- The counit $\epsilon\colon f^*f_* \Rightarrow \mathrm{id}_{\mathcal{E}}$ is monic.

Since f^* for a hyperconnected geometric morphism $f\colon \mathcal{E} \to \mathcal{F}$ is fully faithful, the codomain topos $\mathcal{F}$ can be regarded as a full subcategory of the domain topos $\mathcal{E}$. If we regard it as a full subcategory, we call it a ***hyperconnected quotient***, according to the terminology in [Law07, section VII].

Proposition 2.2. *For any Grothendieck topos $\mathcal{E}$, the partially ordered class of equivalence classes of hyperconnected quotients* $\mathrm{HQ}(\mathcal{E})$ *is a small complete lattice.*

2.2. **Local state classifiers.** We recall the description of $\mathrm{HQ}(\mathcal{E})$ given in [Hor24].

Definition 2.3. We adopt the following definitions:

- A ***local state classifier*** of a category $\mathcal{C}$ is the colimit of all monomorphisms
$$\Xi \coloneqq \operatorname{colim}(\mathcal{C}_{\mathrm{mono}} \to \mathcal{C}),$$
where $\mathcal{C}_{\mathrm{mono}}$ denotes the subcategory consisting of all objects and all monomorphisms of $\mathcal{C}$.
- Its associated cocone is denoted by $\{\xi_X\colon X \to \Xi\}_{X\in\mathrm{ob}(\mathcal{C})}$.

Fact 2.4 ([Hor24])**.** *For a Grothendieck topos $\mathcal{E}$, we have the following facts:*

- *$\mathcal{E}$ has a local state classifier Ξ.*
- *Ξ admits the canonical $\wedge$-semilattice structure such that*
$$\begin{tikzcd} & X \times Y \arrow[dl, "\xi_X \times \xi_Y"'] \arrow[dr, "\xi_{X\times Y}"] & \\ \Xi \times \Xi \arrow[rr, "\wedge"] & & \Xi \end{tikzcd}$$
commutes for every pair of objects (X, Y).
- *The complete lattice of internal filters of Ξ, denoted by* $\mathrm{Filt}(\Xi)$*, is isomorphic to* $\mathrm{HQ}(\mathcal{E})$*:*
$$\mathrm{HQ}(\mathcal{E}) \cong \mathrm{Filt}(\Xi).$$

- *For an internal filter $F \rightarrowtail \Xi$, the counit $\epsilon_X : f^* f_* X \rightarrowtail X$ of the associated geometric morphism $f : \mathcal{E} \to \mathcal{F}$ is given by the pullback diagram*

$$\begin{tikzcd} f^* f_* X \arrow[r] \arrow[d, tail, "\epsilon_X"'] \arrow[dr, phantom, "\lrcorner", very near start] & F \arrow[d, tail] \\ X \arrow[r, "\xi_X"] & \Xi. \end{tikzcd}$$

2.3. **The local state classifier of Σ-Set.** This section describes the local state classifier of the topos Σ-**Set**.

Definition 2.5. We introduce the notion of right congruences:

- A ***right congruence*** on Σ^* is an equivalence relation $\sim$ such that for any $u, v, w \in \Sigma^*$, if $u \sim v$ then $uw \sim vw$.
- For two congruences $\sim$ and $\sim'$, the order relation $\sim \leq \sim'$ means that $u \sim v$ implies $u \sim' v$ for any $u, v \in \Sigma^*$.
- The poset of all right congruences on Σ^* and their order is denoted by Ξ.
- The action of $\sim * w$ is defined by $u \ (\sim * w) \ v$ if and only if $wu \sim wv$.

Proposition 2.6. *We have the following properties:*

- *These data make Ξ an internal semilattice in the topos Σ-**Set**. Furthermore, this is the local state classifier [Hor24] of the topos Σ-**Set**.*
- *In other words, the Σ-set Ξ is the colimit of all monomorphisms in Σ-**Set**.*
- *The colimit cocone $\{\xi_X : X = (Q, \delta) \to \Xi\}_{X \in \mathrm{ob}(\Sigma\text{-}\mathbf{Set})}$ is given by*

$$w \ (\xi_X(q)) \ v \quad \iff \quad qw = qv.$$

In particular, the morphism $\xi_{\mathcal{L}} : \mathcal{L} \to \Xi$ captures the notion of Nerode congruence.

Lemma 2.7. *The component of the colimit cocone $\xi_{\mathcal{L}} : \mathcal{L} \to \Xi$ sends a language L to its Nerode congruence $\sim_L$, where*

$$u \sim_L v \quad \iff \quad u^{-1}L = v^{-1}L.$$

Proposition 2.8 ([Hor24] for Σ-**Set**). *There is a bijective correspondence between equivalence classes of hyperconnected geometric morphisms from Σ-**Set** and the internal filters of Ξ, which furthermore correspond to subsets $F \subset \Xi$ such that*

- *F is closed under Σ^*-actions:*

$$\forall {\sim} \in F, \ \forall w \in \Sigma^*, \ (\sim * w) \in F.$$

- *F contains the trivial equivalence and is closed under finite intersections:*

$$\forall {\sim}, {\sim'} \in F, \ (\sim \wedge \sim') \in F.$$

- *F is upward closed:*

$$\forall {\sim} \in F, \ \forall {\sim'} \in \Xi, \ (\sim \leq \sim' \implies \sim' \in F).$$

For a hyperconnected geometric morphism $h : \Sigma\text{-}\mathbf{Set} \to \mathcal{E}$, let F_h denote the corresponding internal filter of Ξ.

Example 2.9. For the hyperconnected geometric morphism $h : \Sigma\text{-}\mathbf{Set} \to \Sigma\text{-}\mathbf{Set}_{\mathrm{o.f.}}$, the corresponding filter F_h is the set of all congruences $\sim$ whose quotient set $\Sigma^*/\sim$ is finite.

3. Language recognition and syntactic topoi

3.1. Language recognition by a hyperconnected geometric morphism.

Proposition 3.1 (Equivalent definitions of recognition)**.** *For a hyperconnected geometric morphism $h\colon \Sigma\text{-}\mathbf{Set} \to \mathcal{E}$ and a language $L \in \mathcal{L}$, the following conditions are equivalent:*

1. *L is an element of $h^*h_*\mathcal{L} \subset \mathcal{L}$.*
2. *The corresponding filter F_h contains the Nerode congruence $\sim_L$ of L.*

Proof. By Fact 2.4, we have the pullback diagram

$$\begin{tikzcd} h^*h_*\mathcal{L} \arrow[r] \arrow[d, tail, "\epsilon_{\mathcal{L}}"] \arrow[dr, phantom, "\lrcorner", very near start] & F_h \arrow[d, tail] \\ \mathcal{L} \arrow[r, "\xi_{\mathcal{L}}"] & \Xi. \end{tikzcd}$$

This and Lemma 2.7 prove the equivalence. □

Definition 3.2. We say that a hyperconnected geometric morphism $h\colon \Sigma\text{-}\mathbf{Set} \to \mathcal{E}$ ***recognizes*** a language $L \in \mathcal{L}$ if it satisfies the equivalent conditions of Proposition 3.1. The set of languages recognized by h is denoted by $\mathbb{L}_h$.

The set $\mathbb{L}_h$ is the sub-Σ-set $\mathbb{L}_h = h^*h_*\mathcal{L} \rightarrowtail \mathcal{L}$ constructed by the pullback diagram

$$\begin{tikzcd} \mathbb{L}_h = h^*h_*\mathcal{L} \arrow[r] \arrow[d, tail, "\epsilon_{\mathcal{L}}"] \arrow[dr, phantom, "\lrcorner", very near start] & F_h \arrow[d, tail] \\ \mathcal{L} \arrow[r, "\xi_{\mathcal{L}}"] & \Xi. \end{tikzcd}$$

3.2. The syntactic Grothendieck topos associated with a language class.

Definition 3.3. A ***language class*** is a subset $C \subset \mathcal{L}$ of the set of languages $\mathcal{L}$.

Definition 3.4. For a language class $C \subset \mathcal{L}$, its ***syntactic Grothendieck topos*** $\mathcal{T}(C)$ is the minimum hyperconnected quotient that recognizes all languages in C.

For a language $L \in \mathcal{L}$, we call $\mathcal{T}(\{L\})$ the syntactic topos of L and write $\mathcal{T}(L)$ for $\mathcal{T}(\{L\})$ when no confusion can arise.

Remark 3.5 ($\mathcal{T}(C)$ is a geometric morphism)**.** Rigorously speaking, $\mathcal{T}(C)$ is not just a topos, but a topos equipped with the canonical hyperconnected geometric morphism $h\colon \Sigma\text{-}\mathbf{Set} \twoheadrightarrow \mathcal{T}(C)$.

Lemma 3.6. *For any family of right congruences $\{\sim_\lambda\}_{\lambda\in\Lambda}$, there exists a minimum internal filter that contains $\sim_\lambda$ for every $\lambda \in \Lambda$.*

Proof. Since an internal filter is defined by closure properties listed in Proposition 2.8, an arbitrary intersection of internal filters is again an internal filter. Therefore, the intersection of all internal filters that contain $\{\sim_\lambda\}_{\lambda\in\Lambda}$ is the minimum one. □

Definition 3.7 (Generating an internal filter)**.** For a family of right congruences $\{\sim_\lambda\}_{\lambda\in\Lambda}$, the minimum internal filter that contains them is denoted by

$$\langle \sim_\lambda \mid \lambda \in \Lambda \rangle.$$

Proposition 3.8 (Filter corresponding to $\mathcal{T}(C)$)**.** *For a language class C, the internal filter $F_C \subset \Xi$ corresponding to the hyperconnected quotient $\Sigma\text{-}\mathbf{Set} \twoheadrightarrow \mathcal{T}(C)$ is given by*

$$F_C = \langle \xi_{\mathcal{L}}(L) \mid L \in C \rangle.$$

Proof. By Proposition 3.1, the corresponding filter F_C is the minimum internal filter that contains $\xi_{\mathcal{L}}(L)$ for every $L \in C$. $\square$

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