LOCAL STATE CLASSIFIERS RELATIVE TO FACTORISATION SYSTEMS

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Abstract. The notion of a local state classifier was defined to be a colimit of all monomorphisms, which was originally introduced in the topos-theoretic context. This note aims to demystify the fact that such a large colimit exists in many "traditional categories," including $\mathbf{Set}, \mathbf{Grp}, \mathbf{Ring}, \mathbf{Top}$, and $\mathbf{Sh}(\mathcal{C}, J)$, by providing a simple and conceptual existence theorem.

The key observation is that a local state classifier is closely related to two classical notions in category theory: orthogonal factorization systems and total categories. First, some calculations of a local state classifier are naturally generalized to a colimit of the right class of an orthogonal factorization system (from the epi-mono factorization system in a topos). Second, the original sheaf-theoretic construction of a local state classifier works in an arbitrary *total* category, i.e., a category whose Yoneda embedding admits a left adjoint.

Contents

1. Introduction: Local state classifier and its construction problem

In [Hor24], the notion of a local state classifier was introduced. For a category $\mathcal{C}$, we write $\mathcal{C}_{\mathrm{mono}}$ for the subcategory of $\mathcal{C}$ consisting of all objects and all monomorphisms of $\mathcal{C}$.

Definition 1.1 ([Hor24, Definition 3.5]). A **local state classifier** Ξ of a category $\mathcal{C}$ is the colimit of all monomorphisms if it exists.

$$\Xi \coloneqq \operatorname{colim}(\mathcal{C}_{\mathrm{mono}} \rightarrowtail \mathcal{C})$$

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Even if $\mathcal{C}$ is small-cocomplete, $\mathcal{C}$ might not admit a local state classifier, since the indexing category $\mathcal{C}_{\mathrm{mono}}$ is not small. In the paper [Hor24], the author proved that any Grothendieck topos admits a local state classifier by concretely constructing it.

Proposition 1.2 ([Hor24, Proposition 3.21])**.** *For a small site $(\mathcal{C}, J)$, the local state classifier Ξ of the sheaf topos $\mathbf{Sh}(\mathcal{C}, J)$ is given by the sheafification $\Xi = \mathbf{a}\Xi_0$ of the presheaf*

$$\Xi_0 \colon c \mapsto \{\textit{quotient objects of } \mathbf{a}Y(c)\},$$

where $\mathbf{a}\colon \mathbf{PSh}(\mathcal{C}) \to \mathbf{Sh}(\mathcal{C}, J)$ denotes the sheafification functor and $Y : \mathcal{C} \to \mathbf{PSh}(\mathcal{C})$ denotes the Yoneda embedding.

However, this construction is not fully satisfactory in the sense that it is too technical and does not provide any conceptual understanding. The aim of this paper is to demystify the construction by clarifying connections with two classical categorical notions:

- total categories (Section 2), and
- orthogonal factorization systems (Section 3).

In the following sections, we will recall them one by one.

2. Total categories

This section aims to recall the definition and basic properties of total categories. Nothing in this section is a new result.

2.1. Definition and examples.

Notation 2.1. In this paper, **Set** denotes the category of all small sets and functions. For any (possibly large) category $\mathcal{C}$, $\mathbf{PSh}(\mathcal{C})$ denotes the functor category from $\mathcal{C}^{\mathrm{op}}$ to **Set**[1].

Definition 2.2. memo: See e.g. [**street1978yoneda; tholen1980note; street1984family**] A locally small category $\mathcal{C}$ is said to be **total** if its Yoneda embedding $Y_{\mathcal{C}} \colon \mathcal{C} \hookrightarrow \mathbf{PSh}(\mathcal{C})$ admits a left adjoint $L\colon \mathbf{PSh}(\mathcal{C}) \to \mathcal{C}$.

$$\begin{tikzcd} \mathcal{C} \arrow[r, hook, shift right, "Y_{\mathcal{C}}"'] \arrow[r, phantom, "\perp"] & \mathbf{PSh}(\mathcal{C}) \arrow[l, dashed, shift right, "\exists L"'] \end{tikzcd}$$

In this note, 'total category' always means 'locally small total category.'

> [**street1978yoneda**] Total categories are precisely the categories, algebraic and topological, at which traditional category theory was aimed.

Example 2.3 (Easiest example)**.** The easiest category for which one can check the totality is the terminal category **1**. The Yoneda embedding

$$Y_{\mathbf{1}} \colon \mathbf{1} \hookrightarrow \mathbf{PSh}(\mathbf{1}) \cong \mathbf{Set}$$

admits a left adjoint, which is given by the unique functor $L\colon \mathbf{Set} \to \mathbf{1}$.

Example 2.4 (Complete lattice)**.** Every small complete lattice P is total. In fact, one can show that the functor

$$L\colon \mathbf{PSh}(P) \to P \colon F \mapsto \sup\{p \in P \mid F(p) \neq \emptyset\}$$

is a left adjoint to the Yoneda embedding.

[1] Technically, one can take two Grothendieck universes $u_0 \in u_1$. We write **Set** for the u_1-small category of u_0-small sets. Then, for any u_1-small category $\mathcal{C}$, $\mathbf{PSh}(\mathcal{C})$ denotes the functor category $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$, i.e., the u_1-small category of u_0-small set valued presheaves on $\mathcal{C}$.

Example 2.5 (Presheaf categories)**.** The first non-small example of a total category is a presheaf category. Let J be an arbitrary small category. Then, one can prove that the pre-composition functor $-\circ Y_J^{\mathrm{op}} \colon \mathbf{PSh}(\mathbf{PSh}(J)) \to \mathbf{PSh}(J)$ is a left adjoint to the Yoneda embedding functor $Y_{\mathbf{PSh}(J)} \colon \mathbf{PSh}(J) \hookrightarrow \mathbf{PSh}(\mathbf{PSh}(J))$

$$\mathbf{PSh}(J) \underset{Y_{\mathbf{PSh}(J)}}{\overset{-\circ Y_J^{\mathrm{op}}}{\underset{\perp}{\leftrightarrows}}} \mathbf{PSh}(\mathbf{PSh}(J)). \tag{1}$$

(In the case $J = \mathbf{1}$, this is a part of the adjoint 5-tuple in [RW94].)

Example 2.6 (Reflective subcategory of a total category)**.** If a category $\mathcal{C}$ is a reflective subcategory of a total category $\mathcal{D}$

$$\mathcal{C} \underset{\iota}{\overset{\mathbf{a}}{\underset{\perp}{\leftrightarrows}}} \mathcal{D},$$

then we have a chain of adjunction

$$\mathcal{C} \underset{\iota}{\overset{\mathbf{a}}{\underset{\perp}{\leftrightarrows}}} \mathcal{D} \underset{Y_{\mathcal{D}}}{\overset{L}{\underset{\perp}{\leftrightarrows}}} \mathbf{PSh}(\mathcal{D}) \underset{-\circ\iota^{\mathrm{op}}}{\overset{-\circ\mathbf{a}^{\mathrm{op}}}{\underset{\perp}{\leftrightarrows}}} \mathbf{PSh}(\mathcal{C}).$$

Furthermore, its right adjoint part sends an object $c \in \mathrm{ob}(\mathcal{C})$ to a presheaf $\mathcal{D}(\iota-, \iota c)$, which is naturally isomorphic to $\mathcal{C}(-, c) = Y_{\mathcal{C}}(c)$ since ι is fully faithful. This proves that the Yoneda embedding $Y_{\mathcal{C}}$ admits a left adjoint

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift right, hook, "\iota"'] \arrow[r, phantom, "\perp"] \arrow[rrr, bend right, "Y_{\mathcal{C}}"'] & \mathcal{D} \arrow[l, shift right, "\mathbf{a}"'] \arrow[r, shift right, hook, "Y_{\mathcal{D}}"'] \arrow[r, phantom, "\perp"] & \mathbf{PSh}(\mathcal{D}) \arrow[l, shift right, "L"'] \arrow[r, shift right, "-\circ\iota^{\mathrm{op}}"'] \arrow[r, phantom, "\perp"] & \mathbf{PSh}(\mathcal{C}), \arrow[l, shift right, "-\circ\mathbf{a}^{\mathrm{op}}"'] \end{tikzcd} \tag{2}$$

which means that $\mathcal{C}$ is total.

Example 2.7. memo: cite It is known that any reflective subcategory of a total category is total, and that any presheaf category is total. Therefore, all locally presentable categories, including all Grothendieck topoi and all algebraic categories, are total. Other examples include **Top**.

2.2. **Small and large colimits in a total category.** Since a (possibly large) colimit in a reflective subcategory is given by the reflector functor, the calculation of a colimit of $J \xrightarrow{F} \mathcal{C}$ is reduced to that of $J \xrightarrow{F} \mathcal{C} \xrightarrow{Y_{\mathcal{C}}} \mathbf{PSh}(\mathcal{C})$ for a total category $\mathcal{C}$.

Lemma 2.8. *For any locally small category $\mathcal{C}$ and any (possibly large) diagram $F \colon J \to \mathcal{C}$, the colimit of $J \xrightarrow{F} \mathcal{C} \xrightarrow{Y_{\mathcal{C}}} \mathbf{PSh}(\mathcal{C})$ is given by the presheaf*

$$\pi_0(- \downarrow F) \colon c \mapsto \pi_0(c \downarrow F)$$

if $\pi_0(c \downarrow F)$ is small for each $c \in \mathcal{C}$.

Proof. A colimit in a functor category is given by objectwise colimits as soon as the objectwise colimits exist. Therefore, our problem is reduced to the calculation of the colimit of the functor

$$\begin{tikzcd} J \arrow[r, "F"] & \mathcal{C} \arrow[r, "Y_{\mathcal{C}}"] \arrow[rr, bend right, "\mathcal{C}(c,-)"'] & \mathbf{PSh}(\mathcal{C}) \arrow[r, "\mathrm{ev}_c"] & \mathbf{Set}. \end{tikzcd}$$

In general, the colimit of a functor $G \colon J \to \mathbf{Set}$ is given by $\pi_0\left(\int G\right)$ if it is small. This completes the proof since $\int \mathcal{C}(c, -) \circ F = c \downarrow F$. $\square$

Example 2.9 (Final functors and the colimit of Yoneda)**.** If $F\colon J \to \mathcal{C}$ is final (i.e., $\pi_0(c \downarrow F)$ is singleton), then the colimit of $J \xrightarrow{F} \mathcal{C} \xrightarrow{Y_{\mathcal{C}}} \mathbf{PSh}(\mathcal{C})$ is given by the terminal presheaf. In particular, if $F = \mathrm{id}_{\mathcal{C}}\colon J = \mathcal{C} \to \mathcal{C}$, the colimit of the Yoneda embedding is the terminal presheaf.

Proposition 2.10 (colimits in total categories)**.** *For any total category $\mathcal{C}$ with $L \dashv Y_{\mathcal{C}}$ and any (possibly large) diagram $F\colon J \to \mathcal{C}$, if the comma category $c \downarrow F$ has a small number of connected components for each $c \in \mathcal{C}$, then F admits a colimit, which is given by*

$$\operatorname*{colim}_{J} F = L\left(\pi_0(c \downarrow F)\right).$$

3. Orthogonal factorization system and its local state classifier

3.1. **Definition and examples.** This subsection aims to recall the basic properties of orthogonal factorization systems.

Definition 3.1. An **orthogonal factorization system** on a category $\mathcal{C}$ is a pair $(\mathbf{E}, \mathbf{M})$ of subclasses $\mathbf{E}, \mathbf{M} \subset \mathrm{Mor}(\mathcal{C})$ that satisfies the following conditions.

- Both $\mathbf{E}, \mathbf{M}$ contains all isomorphisms.
- Both $\mathbf{E}, \mathbf{M}$ are closed under compositions.
- Any morphism in $\mathcal{C}$ is uniquely decomposed into a morphism in $\mathbf{E}$ followed by one in $\mathbf{M}$ up to unique isomorphisms.

Example 3.2 (Trivial example)**.** For any category $\mathcal{C}$, the pair $(\mathbf{E} = \mathrm{Mor}, \mathbf{M} = \mathrm{Iso})$ is an orthogonal factorization system. Dually, the pair $(\mathbf{E} = \mathrm{Iso}, \mathbf{M} = \mathrm{Mor})$ is also an orthogonal factorization system.

Example 3.3 (Regular categories, topoi, abelian categories, and categories of algebras)**.** For any regular category $\mathcal{C}$, the pair $(\mathbf{E} = \text{regular epi}, \mathbf{M} = \text{mono})$ is an orthogonal factorization system. In particular,

- for any elementary topos $\mathcal{C}$, we have epi-mono factorization system,
- for any abelian category $\mathcal{C}$, we have epi-mono factorization system, and
- for any category of models of an algebraic theory $\mathcal{C}$, we have $(\text{Surj. hom.}, \text{inj. hom.})$-factorization system.

Example 3.4 (quasitopoi)**.**

Example 3.5 (Locally presentable categories.)**.** For any locally presentable category $\mathcal{C}$, the pair $(\mathbf{E} = \text{strong epi}, \mathbf{M} = \text{mono})$ is an orthogonal factorization system. ([AR94])

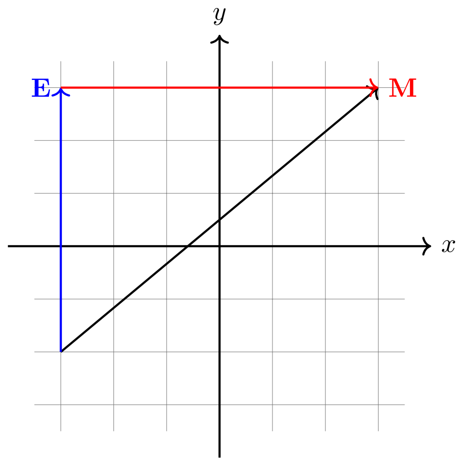
Example 3.6 (Graphical example)**.** Let $\mathbb{R}^2$ denote the poset (since a category) whose objects are elements of $\mathbb{R}^2$ and morphisms are inequalities in the product poset $(\mathbb{R}, \leq)^2$. Then the following classes

- $\mathbf{E} = \{(x, y) \leq (x', y') \mid x = x'\}$: vertical ones
- $\mathbf{M} = \{(x, y) \leq (x', y') \mid y = y'\}$: horizontal ones

define an orthogonal factorization system on the category $\mathbb{R}^2$. Trivially, both classes contain all isomorphisms, which are all identities, and closed under compositions. The unique decomposition of $f : (x, y) \leq (x', y')$ is given by $(x, y) \underset{\mathbf{E}}{\leq} (x, y') \underset{\mathbf{M}}{\leq} (x', y')$ (Theorem 3.6).

For the later reference, we recall some basic properties of orthogonal factorization systems.

Lemma 3.7. *For any category $\mathcal{C}$ equipped with an orthogonal factorization system $(\mathbf{E}, \mathbf{M})$, $\mathbf{E}$ is pushout stable and $\mathbf{M}$ is pullback stable.*



3.2. **Local state classifier relative to factorization system.**

Notation 3.8. For an orthogonal factorization system $(\mathbf{E}, \mathbf{M})$ on a category $\mathcal{C}$, the wide subcategory of $\mathcal{C}$ consisting of all morphisms in $\mathbf{E}$ (resp. $\mathbf{M}$) is denoted by $\mathcal{C}_{\mathbf{E}}$ (resp. $\mathcal{C}_{\mathbf{M}}$).

Definition 3.9. For a category $\mathcal{C}$ equipped with an orthogonal factorization system $(\mathbf{E}, \mathbf{M})$, the **M-local state classifier** is the colimit of the embedding functor $\iota_{\mathbf{M}} \colon \mathcal{C}_{\mathbf{M}} \rightarrowtail \mathcal{C}$, if it exists.

Example 3.10. If $\mathbf{M}$ is the class of all monomorphisms, then the $\mathbf{M}$-local state classifier is the local state classifier in [Hor24].

Example 3.11. For any category $\mathcal{C}$, the pair $(\mathbf{E} = \mathrm{Iso}, \mathbf{M} = \mathrm{Mor})$ is an orthogonal factorization system. The Mor-local state classifier is the colimit of the identity functor, which is known to be the same as the terminal object of $\mathcal{C}$ (See [Rie17] and [**menni2025nonsingular**]). Therefore, a category $\mathcal{C}$ admits a Mor-local state classifier if and only if $\mathcal{C}$ admits a terminal object.

4. Existence theorem

4.1. **The presheaf of quotient objects and its univesality.** Given that theorem 2.10 holds, in order to know $\Xi = \mathrm{colim}(\iota_{\mathbf{M}} \colon \mathcal{C}_{\mathbf{M}} \rightarrowtail \mathcal{C})$, what we need to know is $\pi_0 \left(c \downarrow \iota_{\mathbf{M}}\right)$. Here, we can utilize the structure of orthogonal factorization system.

Definition 4.1. Let $(\mathbf{E}, \mathbf{M})$ be an orthogonal factorization system on a category $\mathcal{C}$. For each $c \in \mathrm{ob}(\mathcal{C})$, let $\mathrm{Quo}_{\mathbf{E}}(c)$ denote the class of all isomorphism classes of $c/\mathcal{C}_{\mathbf{E}}$.

Notation 4.2. Assuming that $\mathcal{C}$ is **E-cowell-powered**, which means that $\mathrm{Quo}_{\mathbf{E}}(c)$ is small for any $c \in \mathrm{ob}(\mathcal{C})$, we define a functor

$$\mathrm{Quo}_{\mathbf{E}} \colon \mathcal{E}^{\mathrm{op}} \to \mathbf{Set}$$

using the $(\mathbf{E}, \mathbf{M})$-factorization (memo: write it later).

Lemma 4.3. *For any category $\mathcal{C}$ equipped with an orthogonal factorization system $(\mathbf{E}, \mathbf{M})$ and an object $c \in \mathrm{ob}(\mathcal{C})$, we have a natural bijection between two classes $\pi_0 \left(c \downarrow \iota_{\mathbf{M}}\right)$ and $\mathrm{Quo}_{\mathbf{E}}(c)$.*

Proof. For any morphism $f \colon c \to d$ in $\mathcal{C}$, we define $q_f \in \mathrm{Quo}_{\mathbf{E}}(c)$ as (the isomorphism class of) the $\mathbf{E}$-part of its $(\mathbf{E}, \mathbf{M})$-factorization. This function $q_{-} \colon \mathrm{ob}\left(c \downarrow \iota_{\mathbf{M}}\right) \to \mathrm{Quo}_{\mathbf{E}}(c)$ is well-defined since the factorization is unique up to a (unique) isomorphism.

We prove that this correspondence defines a well-defined function $q_{-}\colon \pi_0\left(c \downarrow \iota_{\mathbf{M}}\right) \to \mathrm{Quo}_{\mathbf{E}}(c)$ and that it is bijective. To prove that the function is well-defined, what we need to prove is $q_f = q_{m\circ f}$ for any $m \in \mathbf{M}$. This follows since $\mathbf{M}$ is closed under composition and $(\mathbf{E},\mathbf{M})$-factorization is unique up to isomorphisms. The surjectivity follows since for any $e \in \mathbf{E}$, q_e is the isomorphism class of e. Finally, we prove the injectivity. Suppose that $f_0\colon c \to d_0$ and $f_1\colon c \to d_1$ are sent to the same element $q_{f_0} = q_{f_1} \in \mathrm{Quo}_{\mathbf{E}}(c)$. Then we obtain the following commutative diagram

$$\begin{tikzcd}
& \bullet \arrow[r, tail] \arrow[d, "\cong"', "\sigma"] & d_0 \\
c \arrow[ur, two heads] \arrow[dr, two heads] \arrow[urr, bend left, "f_0"] \arrow[drr, bend right, "f_1"'] & & \\
& \bullet \arrow[r, tail] & d_1,
\end{tikzcd}$$

where σ is the unique isomorphism. Since $\mathbf{M}$ contains all isomorphisms, this proves that f_0 and f_1 are in the same connected component in $c \downarrow \iota_{\mathbf{M}}$. $\square$

Remark 4.4 (Easier proof of Theorem 4.3 for proper cases)**.** In practical examples, theorem 4.3 is more easily and conceptually proven. An orthogonal factorization system $(\mathbf{E},\mathbf{M})$ is called **proper** if both $\mathbf{E} \subset \mathrm{Epi}$ and $\mathbf{M} \subset \mathrm{Mono}$ hold. If the considered orthogonal factorization system is proper[2], which often is the case in practical situations, then the functor $\mathcal{C}(c, \iota_{\mathbf{M}}-)\colon \mathcal{C}_{\mathbf{M}} \rightarrowtail \mathcal{C} \to \mathbf{Set}$ is decomposed into the coproduct of representables as follows:

$$\mathcal{C}(c, \iota_{\mathbf{M}}-) \cong \coprod_{[q\colon c \twoheadrightarrow d] \in \mathrm{Quo}_{\mathbf{E}}(c)} \mathcal{C}_{\mathbf{M}}(d, -).$$

In particular, the comma category $c \downarrow \iota_{\mathbf{M}}$ is just the coproduct of slice categories:

$$c \downarrow \iota_{\mathbf{M}} = \int \mathcal{C}(c, \iota_{\mathbf{M}}-) = \coprod_{[q\colon c \twoheadrightarrow d] \in \mathrm{Quo}_{\mathbf{E}}(c)} \int \mathcal{C}_{\mathbf{M}}(d, -) = \coprod_{[q\colon c \twoheadrightarrow d] \in \mathrm{Quo}_{\mathbf{E}}(c)} d/\mathcal{C}_{\mathbf{M}}$$

Its set of connected components obviously coincides with $\mathrm{Quo}_{\mathbf{E}}(c)$, since every slice category is connected.

Proposition 4.5 (Universality of Quo)**.** *For any locally small category $\mathcal{C}$ equipped with an $\mathbf{E}$-cowell-powered orthogonal factorization system $(\mathbf{E},\mathbf{M})$, the presheaf $\mathrm{Quo}_{\mathbf{E}}\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ provides the colimit of the functor $\mathcal{C}_{\mathbf{M}} \rightarrowtail \mathcal{C} \xhookrightarrow{Y} \mathbf{PSh}(\mathcal{C})$.*

$$\mathrm{Quo}_{\mathbf{E}} = \mathrm{colim}(\mathcal{C}_{\mathbf{M}} \rightarrowtail \mathcal{C} \xhookrightarrow{Y} \mathbf{PSh}(\mathcal{C}))$$

Proof. This follows from Theorem 2.10 and Theorem 4.3. Technically, one needs to check that the right $\mathcal{C}$-actions in the two presheaves $\mathrm{Quo}_{\mathbf{E}}$ and $\pi_0\left(c \downarrow \iota_{\mathbf{M}}\right)$ are equal. This is done by observing that the action of a morphism $f\colon c_0 \to c_1$ is given by the π_0 of the pre-composition functor $\left(c_1 \downarrow \iota_{\mathbf{M}}\right) \xrightarrow{-\circ f} \left(c_0 \downarrow \iota_{\mathbf{M}}\right)$. memo: I will write more details on this point. $\square$

[2]Actually, we only uses $\mathbf{E} \subset \mathrm{Epi}$ here.

4.2. Exsitence and construction of a local state classifier.

Theorem 4.6 (Ξ is the best approximation to the collection of quotients)**.** *For any total category $\mathcal{C}$ equipped with an $\mathbf{E}$-cowell-powered factorization system $(\mathbf{E}, \mathbf{M})$, $\mathcal{C}$ admits an $\mathbf{M}$-local state classifier $\Xi_{\mathbf{M}}$. Furthermore, it is given by*

$$\Xi_{\mathbf{M}} = L(\mathrm{Quo}_{\mathbf{E}}),$$

where $L\colon \mathbf{PSh}(\mathcal{C}) \to \mathcal{C}$ denotes the left adjoint to the Yoneda embedding.

Corollary 4.7. *Let $\mathcal{C}$ be a reflective subcategory of a presheaf category $\mathbf{PSh}(J)$ for a small category J*

$$\begin{tikzcd} \mathcal{C} \arrow[r, hook, shift right, "\iota"'] \arrow[r, phantom, "\perp"] & \mathbf{PSh}(J). \arrow[l, shift right, "\mathbf{a}"'] \end{tikzcd}$$

Then, for any $\mathbf{E}$-cowell-powered orthogonal factorization system $(\mathbf{E}, \mathbf{M})$ on $\mathcal{C}$, the $\mathbf{M}$-local state classifier $\Xi_{\mathbf{M}}$ exists. Furthermore, it is given by applying $\mathbf{a}$ to the presheaf $J^{\mathrm{op}} \to \mathbf{Set} : j \mapsto \mathrm{Quo}_{\mathbf{E}}(\mathbf{a}Y_J(j))$:

$$\Xi_{\mathbf{M}} = \mathbf{a}\left(J^{\mathrm{op}} \xrightarrow{Y_J^{\mathrm{op}}} \mathbf{PSh}(J)^{\mathrm{op}} \xrightarrow{\mathbf{a}^{\mathrm{op}}} \mathcal{C}^{\mathrm{op}} \xrightarrow{\mathrm{Quo}_{\mathbf{E}}} \mathbf{Set}\right).$$

Proof. Combining two adjunctions 1 and 2, we have a combined adjunction

$$\begin{tikzcd} \mathcal{C} \arrow[r, hook, shift right, "\iota"'] \arrow[r, phantom, "\perp"] \arrow[rrr, hook, bend right, "Y_{\mathcal{C}}"'] & \mathbf{PSh}(J) \arrow[l, shift right, "\mathbf{a}"'] \arrow[r, hook, shift right, "Y_{\mathbf{PSh}(J)}"'] \arrow[r, phantom, "\perp"] & \mathbf{PSh}(\mathbf{PSh}(J)) \arrow[l, shift right, "-\circ Y_J^{\mathrm{op}}"'] \arrow[r, shift right, "-\circ\iota^{\mathrm{op}}"'] \arrow[r, phantom, "\perp"] & \mathbf{PSh}(\mathcal{C}). \arrow[l, hook, shift right, "-\circ\mathbf{a}^{\mathrm{op}}"'] \end{tikzcd}$$

$\square$

Notice that this corollary subsumes Theorem 1.2, while this new version is much more easily proven.

Corollary 4.8. *Every locally presentable category has a local state classifier.*

Proof. Any locally presentable category admits the $(\mathrm{StrongEpi}, \mathrm{Mono})$-factorization system. It is also total and cowell-powered. $\square$

5. Adding legs to a snake: Classification theorems

5.1. Classification of M-pullback stable family of M-subobjects. memo: Consider M-subobject classifier, or just $\mathrm{Quo}_{\mathbf{E}} \to \mathrm{Sub}_{\mathbf{M}}$

Notation 5.1. For a category $\mathcal{C}$ equipped with an $\mathbf{M}$-powered orthogonal factorization system $(\mathbf{E}, \mathbf{M})$, assuming that $\mathcal{C}$ has pullbacks along $\mathbf{M}$-morphisms, we define

$$\mathrm{Sub}_{\mathbf{M}}\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$$

as the presheaf that sends an object $c \in \mathrm{ob}(\mathcal{C})$ to the set of all isomorphism classes of $\mathcal{C}_{\mathbf{M}}/c$.

Remark 5.2 ($\mathrm{Sub}_{\mathbf{M}}$ is NOT dual to $\mathrm{Quo}_{\mathbf{E}}$.). $\mathrm{Sub}_{\mathbf{M}}$ is not $\mathrm{Quo}_{\mathbf{E}}$ for the opposite category $\mathcal{C}^{\mathrm{op}}$ with the opposite factorization system $(\mathbf{M}^{\mathrm{op}}, \mathbf{E}^{\mathrm{op}})$. In fact, $\mathrm{Quo}_{\mathbf{M}^{\mathrm{op}}}\colon \mathcal{C} = \mathcal{C}^{\mathrm{opop}} \to \mathbf{Set}$ is a covariant (not contravariant!) functor, which has the same object function $c \mapsto \mathrm{Quo}_{\mathbf{M}^{\mathrm{op}}}(c) = \mathrm{Sub}_{\mathbf{M}}(c)$. In the case of $\mathcal{C} = \mathbf{Set}$, $\mathrm{Quo}_{\mathbf{M}^{\mathrm{op}}}$ is the covariant powerset functor, and $\mathrm{Sub}_{\mathbf{M}}$ is the contravariant powerset functor.

Proposition 5.3. *For any category $\mathcal{C}$ equipped with an $\mathbf{E}$-cowell-powered and $\mathbf{M}$-well-powered orthogonal factorization system $(\mathbf{E}, \mathbf{M})$ and with $\mathbf{M}$-pullbacks, there is a one-to-one correspondence between*

- *natural transformations $\mathrm{Quo}_{\mathbf{E}} \to \mathrm{Sub}_{\mathbf{M}}$ in $\mathbf{PSh}(\mathcal{C})$ and*

- *families of* $\mathbf{M}$*-subobjects* $\{S_c\}_{c\in \mathrm{ob}(\mathcal{C})}$ *that is stable under* $\mathbf{M}$*-pullbacks.*

Proof. This is an immediate corollary of Theorem 4.5 and the Yoneda lemma. ☐

Corollary 5.4. *In the situation of Theorem 5.3, if* $\mathcal{C}$ *is total and* $\mathrm{Sub}_{\mathbf{M}}$ *is representable* $\mathrm{Sub}_{\mathbf{M}} \cong \mathcal{C}(-, \Omega_{\mathbf{M}})$*, then we can add*

- *morphisms* $\Xi_{\mathbf{M}} \to \Omega_{\mathbf{M}}$ *and*
- $\mathbf{M}$*-subobjects of* $\Xi_{\mathbf{M}}$*.*

to the correspondence in Theorem 5.3.

Proof. We have $\mathbf{PSh}(\mathcal{C})(\mathrm{Quo}_{\mathbf{E}}, \mathrm{Sub}_{\mathbf{M}}) = \mathbf{PSh}(\mathcal{C})(\mathrm{Quo}_{\mathbf{E}}, Y_{\mathcal{C}}(\Omega_{\mathbf{M}})) \cong \mathcal{C}(L(\mathrm{Quo}_{\mathbf{E}}), \Omega_{\mathbf{M}}) = \mathcal{C}(\Xi_{\mathbf{M}}, \Omega_{\mathbf{M}}) \cong \mathrm{Sub}_{\mathbf{M}}(\Xi_{\mathbf{M}})$ due to $\Xi_{\mathbf{M}} = L(\mathrm{Quo}_{\mathbf{E}})$ (Theorem 4.6). ☐

This corollary subsumes the Grothendieck topos case of the broader correspondence theorem in [Proposition 4.5. Hor24].

Remark 5.5 ($\mathrm{Quo}_{\mathbf{E}}$ is rarely representable)**.** In contrast to the situation that an $\mathbf{M}$-subobject classifier represents the presheaf of subobjects (cf. topos, or quasitopos), the presheaf of quotient objects is rarely representable: If $\mathrm{Quo}_{\mathbf{E}}$ is representable, then there exists an object $Q \in \mathrm{ob}(\mathcal{C})$ such that every object $X \in \mathrm{ob}(\mathcal{C})$ admits an $\mathbf{M}$-morphism into Q. memo: Can we say $\mathbf{M} = \mathcal{C}$?

Remark 5.6. [Ken06]

Example 5.7 (Grothendieck quasitopos)**.**

5.2. Classification with additional structures.

Theorem 5.8 ([**wood1982some**])**.** *A total category* $\mathcal{C}$ *with* $L \dashv Y_{\mathcal{C}}$ *is cartesian closed if and only if* L *preserves binary products.*

Therefore, for a total cartesian closed category $\mathcal{C}$, the $\mathbf{M}$-local state classifier $\Xi_{\mathbf{M}} = L(\mathrm{Quo}_{\mathbf{E}})$ inherits the semilattice structure of $\mathrm{Quo}_{\mathbf{E}}$.

memo: When does it preserve order structure? Related to this question, when is L lex? $= \mathcal{C}$ is a Grothendieck topos! See [**street1981notions**]

Example 5.9 (Grothendieck quasitopoi)**.**

Example 5.10 (local state classifier in a locally presentable category)**.**

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